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THE CONGRUENCE EXTENSION PROPERTY FOR COMPACT TOPOLOGICAL LATTICES

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Let L be a compact, distributive topological lattice of finite breadth and let A be a closed sublattice of L. It is shown that every closed congruence on A can be extended to a closed congruence on L. An example is provided to show that the requirement of finite breadth cannot be deleted.

The congruence extension property serves to characterize distributive lattices (cf. [4]). The definition of this property may be reformulated for topological lattices as follows: A topological lattice L has the congruence extension property if given any closed sublattice A of L and any closed congruence $[\varphi]$ on A there is a closed congrucence $[\Phi]$ on L such that $[\Phi] \cap (A \times A) = [\varphi]$. When this situation prevails we say that $[\varphi]$ has been extended to a closed congruence $[\Phi]$ on L. In this paper we prove that compact, distributive lattices of finite breadth have the congruence extension property. This fact is established by first showing that the lattice of closed congruences for such lattices is distributive. We also prove that the compact topological lattice $X = \times_{i=1}^{\infty}\{0, 1\}$ with coordinatewise operations does not have the congruence extension property.

1. Preliminaries. A finite subset B of a lattice is meet-irredundant if no proper subset of B has the same meet as B. The breadth of a lattice L, $Br(L)$, is the supremum of the cardinalities of its meet-irredundant sets. A chain is a lattice whose breadth is one. An element p of a lattice is prime if $x \wedge y \leqq p$ implies that $x \leqq p$ or $y \leqq p$. We shall use the notation that if $[\varphi]$ is a congruence then φ is the canonical map associated with $[\varphi]$.

A topological lattice is a Hausdorff topological space with a pair of continuous maps $\wedge, \vee : L \times L \to L$ such that $(L, \wedge, \vee)$ is a lattice. A point p of a subset A of a topological lattice L is a local maximum of A if there is an open subset U of L such that $(U \cap A) \cap (p \vee L) = \{p\}$. By A^* we shall mean the topological closure of A.

Suppose that L is a compact topological lattice. $\mathscr{C}(L)$ is the lattice of closed congruences on L (considered as subsets of $L \times L$) with operations $\wedge$ and $\vee$ defined by $[\varphi] \wedge [\theta] = [\varphi] \cap [\theta]$ and $[\varphi] \vee [\theta]$ is the smallest closed congruence on L which contains both $[\varphi]$ and $[\theta]$. $\mathscr{L}(L) = \{[\varphi] \in \mathscr{C}(L); Br(\varphi(L)) = 1\}$. For $[\varphi] \in \mathscr{L}(L)$ we define $M(\varphi) = \{x \in L; x = \vee \varphi^{-1}(\varphi(x))\}$ and $m(\varphi) = \{x \in L; x = \wedge \varphi^{-1}(\varphi(x))\}$.