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ON THE DENSITY OF (k, r) INTEGERS

Y. K. FENG AND M. V. SUBBARAO

Let k and r be integers such that $0 < r < k$. We call a positive integer n, $a(k, r)$-integer if it is of the form $n = a^K b$, where a and b are natural numbers and b is r-free. Clearly, $a(\infty, r)$-integer is a r-free integer. Let $Q_{k,r}$ denote the set of (k, r)-integers and let $\delta(Q_{k,r})$, $D(Q_{k,r})$ respectively denote the asymptotic and Schnirelmann densities of the set $Q_{k,r}$. In this paper, we prove that $\delta(Q_{k,r}) > D(Q_{k,r}) \geqq \zeta(k)(1 - \sum_p p^{-r}) - 1/k(1 - (1/k))^{k-1}$, and deduce the known results for r-free integers.

1. Introduction and Notation. In some recent papers, ([4, 5]) we introduced a generalized class of r-free integers, which we called the (k, r)-integers. For given integers k, r with $0 < r < k$, $a(k, r)$-integer is one whose k-free part is also r-free. In the limiting case when $k = \infty$, we get the r-free integers. It is clear that $a(k, r)$-integer is an integer of the form $a^k b$, where a and b are natural numbers and b is r-free. Let $Q_{k,r}$, Q_r denote the set of all (k, r)-integers and the set of all r-free integers respectively. Also let $Q_{k,r}(x)$ denote the number of (k, r)-integers not exceeding x, with corresponding meaning for $Q_r(x)$. We write $\delta(Q_{k,r})$ for the asymptotic density of the (k, r)-integers, that is,

$$\delta(Q_{k,r}) = \lim_{x\to\infty} \frac{Q_{k\ r}(x)}{x} ,$$

(provided this limit exists), and $D(Q_{k,r})$ for their Schnirelmann density given by

$$D(Q_{k,r}) = \inf_n \frac{Q_{k\ r}(n)}{n} .$$

We define $\delta(Q_r)$ and $D(Q_r)$ analogously. Let $\psi(n)$ be the characteristic function of $Q_{k,r}$ and $\lambda(n)$ be defined by

$$\sum_{d|n} \lambda(d) = \psi(n) .$$

It is easily proved (see [3]) that the function $\psi(n)$ and $\lambda(n)$ are multiplicative and for any prime p

$$\lambda(p^a) = \begin{cases} 1 & a \equiv 0 \pmod{k} , \\ -1 & a \equiv r \pmod{k} , \\ 0 & \text{otherwise.} \end{cases}$$

Further,