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GLOBALIZATION THEOREMS FOR LOCALLY FINITELY GENERATED MODULES

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Each commutative ring has a coreflection $\hat{R}$ in the category of commutative regular rings. We use the basic properties of $\hat{R}$ to obtain globalization theorems for finite generation and for projectivity of R-modules.

1. Preliminaries. A detailed description of the ring $\hat{R}$ may be found in [8]. Here we list without proofs the facts that will be needed. We assume that everything is unitary, but not necessarily commutative. However, R will always denote an arbitrary *commutative* ring. All unspecified tensor products are taken over R. For each $a \in R$ and each $P \in \operatorname{Spec}(R)$, let $a(P)$ be the image of a under the obvious map $R \rightarrow R_P/PR_P$. Then $\hat{R}$ is the subring $\prod_P R_P/PR_P$ consisting of finite sums of elements $[a, b]$, where $[a, b]$ is the element whose P^{th} coordinate is 0 if $b \in P$ and $a(P)/b(P)$ if $b \notin P$. There is a natural homomorphism $\varphi\colon R \rightarrow \hat{R}$ taking a to $[a, 1]$. The ring $\hat{R}$ is regular (in the sense of von Neumann). The statement that $\hat{R}$ is a coreflection means simply that each homomorphism from R into a commutative regular ring factors uniquely through φ.

The map $\operatorname{Spec}(\varphi)\colon \operatorname{Spec}(\hat{R}) \rightarrow \operatorname{Spec}(R)$ is one-to-one and onto; for each $P \in \operatorname{Spec}(R)$ we let $\hat{P}$ be the corresponding prime ($=$ maximal) ideal of $\hat{R}$.

If A is an R-module and $P \in \operatorname{Spec}(R)$, then A_P/PA_P and $(A \otimes \hat{R})_{\hat{P}}$ are vector spaces over R_P/PR_P and $\hat{R}_{\hat{P}}$ respectively. The map $\varphi\colon R \rightarrow \hat{R}$ induces an isomorphism $R_P/PR_P \cong \hat{R}_{\hat{P}}$, and, under the identification, A_P/PA_P and $(A \otimes \hat{R})_{\hat{P}}$ are isomorphic vector spaces.

2. Globalization theorems.

LEMMA. *If $A \otimes \hat{R} = 0$ and A_R is locally finitely generated then $A = 0$.*

Proof. For each prime P, $A_P/PA_P = 0$, by the last paragraph of § 1. Since A_P is finitely generated over R_P, Nakayama's lemma implies that $A_P = 0$ for each $P \in \operatorname{Spec}(R)$. Therefore $A = 0$.

THEOREM 1. *Assume $(A \otimes \hat{R})$ is finitely generated over $\hat{R}$, and that A_R is either locally free or locally finitely generated. Then A_R is finitely generated.*