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ON SOLVABLE O*-GROUPS

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The purpose of this paper is to prove the existence of O*-groups of arbitrary solvable length, as well as of non-solvable O*-groups.

By a *partial order* for a group G we mean a reflexive, antisymmetric and transitive relation, $\leqq$, on G such that if g and h are elements of G and $g \leqq h$, then $xgy \leqq xhy$ for all x and y in G. If also any two elements g and h of G are *comparable* (i.e., either $g \leqq h$ or $h \leqq g$), then the partial order for G is called a *total* (or *full*, or *linear*) *order*. The group G is an *O*-group* if any partial order for G is included in some total order for G.

A group G is *solvable of length* n, where n is a positive integer, if the derived chain of G reaches the unit subgroup, E, in n steps:

$$G = G^1 \supsetneqq G^2 \supsetneqq \cdots \supsetneqq G^n \supsetneqq G^{n+1} = E,$$

where G^{i+1} is the derived group of G^i (denoted below by $G^{i+1} = [G^i, G^i]$).

It has been shown that non-abelian free groups are not O*-groups ([1], [2], [3], [4], [6]). Further, Kargapolov [5] and Kargapolov, Kokorin and Kopytov [6] have produced solvable groups which are not O*-groups even though they admit a full order: these are the free r-step solvable groups on k generators for $r \geqq 3$ and $k \geqq 2$. In view of these results one may ask if there exist solvable O*-groups of arbitrary length, and nonsolvable O*-groups. The answers are affirmative.

THEOREM. *For every positive integer m there exists an O*-group G that is solvable of length m.*

Proof. Let F be the free group on k generators for some fixed $k \geqq 2$. Let F_i be the ith term in the lower central series for F, where $F_1 = F$, and let F^i be the ith derived group for F, where $F^1 = F$. Consider F/F_i, the free nilpotent group of class i with k generators. By varying i we shall obtain the desired groups G of the theorem.

We first claim that F/F_i is torsion-free for every positive integer i. If not, then for some i there exists an element $a \in F$ and a positive integer p such that $a \notin F_i$, but $a^p \in F_i$. Now $a \in F_h - F_{h+1}$ for some positive integer $h \leqq i - 1$. Thus $a^p \in F_i \subseteqq F_{h+1}$, and so F_h/F_{h+1} is not torsion-free. On the other hand, Witt's theorem (see, e.g., [8, p. 41]) states that F_h/F_{h+1} is a free abelian group (and hence torsion-