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ON RANK 3 PROJECTIVE PLANES

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One of the unsolved problems in the theory of projective planes is the following: Is every finite projective plane with a transitive collineation group desarguesian? This problem is investigated under the additional hypothesis that the group has rank 3. It is proven that if a projective plane $\mathscr{P}$ of order $n > 2$ has a rank 3 collineation group then $\mathscr{P}$ is non-desarguesian and either (i) n is odd and $n = m^4$, or (ii) n is even and $n = m^2$ with $m = 0 \pmod 4$.

One of the older problems in the theory of projective planes is the following: If a finite projective plane $\mathscr{P}$ has a collineation group G transitive on its points, is the plane desarguesian? So far only under one or more additional assumptions has the answer been shown to be yes. The basis for these results is the theorem, due to Wagner [10], that if the transitive group G contains a central collineation, then $\mathscr{P}$ is desarguesian.

Ostrom and Wagner [8] showed that if the group G is doubly transitive then $\mathscr{P}$ is desarguesian, and G contains all elations of $\mathscr{P}$. Higman and McLaughlin [4] investigated the problem in the case when the group G is transitive on the flags of $\mathscr{P}$. They proved that under certain restrictions on the order of $\mathscr{P}$ the plane is desarguesian. Keiser [6] and Wagner [10] have showned that under restrictions on the order of $\mathscr{P}$ and the order of G the plane is desarguesian.

The rank of a permutation group G transitive on a set Ω is the number of orbits of G_P, P a point of Ω, in Ω. Hence a transitive group G has rank 2 on a set Ω if and only if G is doubly transitive on Ω. G has rank 3 if and only if for every point $P \in \Omega$ G_P has two orbits besides G_P. Ostrom and Wagner have thus answered the question when the group G has rank 2. It is then natural to ask: If a finite projective plane has a transitive collineation group of rank 3, is the plane desarguesian?

Investigating this question we have found that a more appropriate question is: Which finite projective planes have rank 3 collineation groups? For we will prove in this article the following

MAIN THEOREM. *Let $\mathscr{P}$ be a finite projective plane of order n with rank 3 collineation group G. Then n satisfies one of the statements*:

(i) $n = 2$

(ii) *n is odd and* $n = m^4$