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TWO REMARKS ON ELEMENTARY EMBEDDINGS OF THE UNIVERSE

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The paper contains the following two observations: 1. The existence of the least submodel which admits a given elementary embedding j of the universe. 2. A necessary and sufficient condition on a complete Boolean algebra B that the Cohen extension V^B admits j.

A function j defined on the universe V is an *elementary embedding of the universe* if there is a submodel M such that for any formula φ,

$$(*) \qquad \forall x_1, \cdots, x_n[\varphi(x_1, \cdots, x_n) \longleftrightarrow M \models \varphi(jx_1, \cdots, jx_n)].$$

Let j be an elementary embedding of the universe. If N is a submodel, let $j_N = j \mid N$ be the restriction of j to N. N *admits* j if

$$(**) \qquad N \models j_N \text{ is an elementary embedding of the universe.}$$

If B is a complete Boolean algebra, let V^B be the Cohen extension of V by B. V^B *admits* j if

$$(***) \qquad V^B \models \text{there exists an elementary embedding } i \text{ of the universe such that } i \supseteq j$$

THEOREM 1. *There is a submodel $L(j)$ which is the least submodel which admits j.*[1]

THEOREM 2. *The Cohen extension V^B admits j if and only if the identity mapping on $j''B$ can be extended to a $j(V)$ – complete homomorphism of $j(B)$ onto $j''B$.*

Before giving the proof, we have a few remarks. The underlying set theory is the axiomatic theory BG of sets and classes of Bernays and Gödel [1]. The formula φ in (*) is supposed to have only set variables. However, if for any class C we let $j(C) = \bigcup_{\alpha \in On} j(C \cap V_\alpha)$, then (*) holds also for formulas having free class variables ("normal formulas" of [1].) Incidentally, "j is an elementary embedding of the universe" is expressible in the language of BG (viz.: j is an ε-isomorphism and $\forall C_1 \forall C_2[\mathscr{F}_i(jC_1, jC_2) = j(\mathscr{F}_i(C_1, C_2))]$ where $\mathscr{F}_i$ are the Gödel operations).

[1] This was observed independently by K. Hrbáček, giving a different proof.