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A NOTE ON TWO GENERALIZATIONS OF QF-3

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If M is an R-module, then the dual of M is defined to be $\mathrm{Hom}_R(M, R)$. Artinian QF-3 rings R have been characterized by the following two properties:

(1) The class of R-modules with zero duals is closed under taking submodules.

(2) The class of torsionless R-modules is closed under extension.

These properties are independent and, in the present paper, we study the two classes of rings R which satisfy each of these conditions separately.

Let R be a ring with identity. R is said to be (left) QF–3 provided there is an idempotent e in R such that Re is faithful and injective as a (left) R-module. The notion of QF–3 rings is derived from the definition of QF–3 algebras introduced by Thrall in [4].

If M is a left R-module, let $M^* = \mathrm{Hom}(M, R)$ denote the "dual" of M, with the usual right module structure. For left Artinian rings R, Wu, Mochizuki and Jans [5] have given the following two properties characterizing those which are QF–3.

(1) If $M_1 \subseteq M_2$ are R-modules, then $M_2^* = (0)$ implies $M_1^* = (0)$.

(2) The class of torsionless R-modules is closed under extension.

That is, if A and C are torsionless R-modules, and $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then B is torsionless.

In this note, rings satisfying (1) or (2) separately are studied. Those satisfying (1) are called SZD and those satisfying (2), TCE. For (left) R-modules M, the following notation is used,

$Z(M) = \{m \in M \mid Em = 0$ for some essential left ideal $E \subseteq R\}$ (the singular submodule of M)

$S(M) =$ the sum of all simple submodules of M (the socle of M)

$E(M) =$ injective hull of M

SZD and TCE Rings

PROPOSITION 1. *A ring R is SZD if and only if the following are equivalent for every R-module M.*

(1) $\mathrm{Hom}(M, R) = (0)$ (2) $\mathrm{Hom}(M, E(R)) = (0)$

Proof. Assume R is SZD. Condition (2) implies (1) trivially. To show (1) implies (2), assume $M^* = (0)$ and let $f \neq 0$ in $\mathrm{Hom}(M, E(R))$. Set $L = f(M) \cap R$ and $M_0 = f^{-1}(L)$. Then $M_0 \neq (0)$ and $f|_{M_0}: M_0 \to R$ is nonzero, so that $M_0^* \neq (0)$. Since R is SZD, this implies $M^* \neq (0)$,