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UNKNOTTING CONES IN THE TOPOLOGICAL CATEGORY

R. RICHARD SUMMERHILL

Let Q be a topological q-manifold, let X be a compact metric space, and let bQ and aX denote the cones over Q and X, respectively. A proper embedding $f\colon aX \to bQ$ (i.e., $f(a) = b$ and $f^{-1}[Q] = X$) is unknotted if there is homeomorphism $h\colon bQ \to bQ$ such that $hf = \bar{f}$, where $\bar{f}$ is the conical extension of f. In this paper it is proved that a proper embedding is unknotted if and only if $bQ - f[aX]$ and $bQ - \bar{f}[ax]$ are of the same homotopy type and the embedding f satisfies a local flatness condition.

In this paper we present a topological analog to Lickorish's theorem concerning the PL unknotting of cones [7]. The PL result states that if one embeds the cone over a complex into a ball (with a codimension restriction) such that the base and only the base of the cone sits in the boundary of the ball, then one can deform the ball (without moving the boundary) so as to straighten out the cone. The codimension requirement is that the dimension of the cone be at least three less than the dimension of the ball.

We consider here a similar problem in the topological category where the complex is replaced by a compact metric space and the ball is replaced by the cone over a topological manifold. Homotopy conditions are used instead of codimension, and, of course, some local flatness condition is needed. This condition generalizes that property for manifolds and is defined by using the inherent fibre structure of the cone.

Our main theorem is then: An embedding of the cone over a compact metric space into the cone over a compact topological manifold is unknotted if and only if (1) certain homotopy properties are satisfied and (2) the embedding is "locally flat."

The proof of this theorem follows precisely the same outline as the proof of the unknotting theorem by Price and Glaser (Theorem 1 of [4]), but uses topological engulfing in place of PL engulfing.

1. Definitions. Throughout this paper the term manifold will be used in the topological sense. That is, a q-manifold Q is a separable metric space in which each point has a closed neighborhood homeomorphic to a q-cell. Let BdQ denote the boundary of Q and $\text{Int}Q$ the set Q-BdQ. The manifold Q is *closed* if it is compact and without boundary. We let I denote the closed unit interval $[0, 1]$