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O-PRIMITIVE ORDERED PERMUTATION GROUPS

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Let G be a transitive l-subgroup of the lattice-ordered group $A(\Omega)$ of all order-preserving permutations of a chain Ω. (In fact, many of the results are generalized to partially ordered sets Ω and transitive groups G such that $\beta < \gamma$ implies $\beta g = \gamma$ for some positive $g \in G$, thus encompassing some results on non-ordered permutation groups.) The orbits of any stabilizer subgroup G_α, $\alpha \in \Omega$, are convex and thus can be totally ordered in a natural way. The usual pairing $\Delta \longleftrightarrow \Delta' = \{\alpha g \mid \alpha \in \Delta g\}$ establishes an o-anti-isomorphism between the set of "positive" orbits and the set of "negative" orbits. If Δ is an o-block (convex block) of G for which $\Delta G_\alpha = \Delta$, then Δ' is also an o-block. If G_α has a greatest orbit Γ, then $\{\beta \in \Omega \mid \Gamma' < \beta < \Gamma\}$ constitutes an o-block of G. A correspondence is established between the centralizer $Z_{A(\Omega)}G$ and a certain subset of the fixed points of G_α.

The main theorem states that every o-primitive group (G, Ω) which is not o-2-transitive or regular looks strikingly like the only previously known example, in which Ω is the reals and $G = \{f \in A(\Omega) \mid (\beta + 1)f = \beta f + 1 \text{ for all } \beta \in \Omega\}$. The "configuration" of orbits of G_α must consist of a set o-isomorphic to the integers of "long" (infinite) orbits with some fixed points interspersed; and there must be a "period" $z \in Z_{A(\bar{\Omega})}G$ ($\bar{\Omega}$ the Dedekind completion of Ω) analogous to the map $\beta z = \beta + 1$ in the example. Periodic groups are shown to be l-simple, and more examples of them are constructed.

Transitivity guarantees that the "configuration" of orbits of G_α is independent of α, so that we may speak of the *configuration* of G (defined more precisely later). There is appreciable interplay between this configuration and other properties of G. For example, o-2-transitive groups are characterized by having only one positive orbit, and regular groups by having configurations consisting entirely of fixed points.

For periodically o-primitive groups, the period z is the unique o-permutation of $\bar{\Omega}$ such that for every $\beta \in \Omega$, βz is the sup of the first positive orbit of G_β. $(\beta z)g = (\beta g)z$ for all $\beta \in \Omega$, $g \in G$, and in fact z generates $Z_{A(\bar{\Omega})}G$. This periodicity is of paramount importance. For example, it guarantees that the action of $g \in G$ on any long orbit of G_α determines its action on all of Ω.

Transitive l-subgroups of $A(\Omega)$ have been studied from a lattice-ordered group (l-group) orientation by Holland [5, 6, 7], Lloyd [10, 11], Sik [15], and McCleary [12, 13]. Holland showed that every l-group