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MAXIMINIMAX, MINIMAX, AND ANTIMINIMAX THEOREMS AND A RESULT OF R. C. JAMES

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This paper contains a number of minimax theorems in various topological and non-topological situations. Probably the most interesting is the following: if X is a nonempty bounded convex subset of a real Hausdorff locally convex space E with dual E' and each $\varphi \in E'$ attains its supremum on X then

$$\left.\begin{array}{c}\textbf{for all nonempty convex equicontinuous } Y \subset E' \\ \inf_{y \in Y} \sup \langle X, y\rangle \leqq \sup_{x \in X} \inf \langle x, Y\rangle\end{array}\right\}(*) .$$

It is also proved that if (*) is true and X is complete then X is $w(E, E')$-compact. Combining these results, a proof of a well known result of R. C. James is obtained.

We suppose throughout that $X \neq \phi$, $Y \neq \phi$, and $f: X \times Y \rightarrow R$. We write $\mathscr{F}(X)$ for $\{F: \phi \neq F \subset X, F \text{ is finite}\}$ and define $\mathscr{F}(Y)$ similarly. The *maximinimax inequality* is the relation

$$(1) \qquad \inf_{G \in \mathscr{F}(Y)} \sup_{x \in X} \inf f(x, G) \leqq \sup_{F \in \mathscr{F}(X)} \inf_{y \in Y} \sup f(F, y)$$

and the *minimax inequality* is the relation

$$(2) \qquad \inf_{y \in Y} \sup f(X, y) \leqq \sup_{x \in X} \inf f(x, Y) .$$

The main result of this paper is Theorem 5, which gives some conditions under which (1) holds. These conditions are completely non-topological and depend only on the fact that certain functions attain their suprema on X. We prove Theorem 5 by defining a "remoteness" relation on the subsets of Y, but we point out that Theorem 5 can also be proved by first reducing the problem to the "iterated limits unequal" situation (by using the technique of Remark 8 and then the diagonal process) and then going through the same steps as in [6], Lemmas 1–7. The approach adopted here embodies a new type of diagonal argument (Lemmas 2 and 3) which might find applications elsewhere, and an argument similar to but subtler than that used in [9], Lemma 2. There is another proof of Theorem 5 that is "frontended" in the sense that we can choose the functions $k_1, k_2, \cdots$ of Theorem 5 by a purely inductive process without having first to choose a sequence $\{y_n\}_{n \geqq 1}$. The price one pays for the "frontendedness" is that the induction is more complicated and that is why we have avoided the alternative approach.