PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

NORM CONVERGENCE OF MARTINGALES OF RADON-NIKODYM DERIVATIVES GIVEN A σ-LATTICE

R. B. DARST AND G. A. DEBOTH

Suppose that $\{\mathscr{M}_k\}$ is an increasing sequence of sub σ-lattices of a σ-algebra $\mathscr{A}$ of subsets of a non-empty set Ω. Let $\mathscr{M}$ be the sub σ-lattice generated by $\bigcup_k \mathscr{M}_k$. Suppose that L^Φ is an associated Orlicz space of $\mathscr{A}$-measurable functions, where Φ satisfies the Δ_2-condition, and let $h \in L^\Phi$. It is verified that the Radon-Nikodym derivative, f_k, of h given $\mathscr{M}_k$ is in L^Φ and shown that the sequence $\{f_k\}$ converges to f in L^Φ, where f is the Radon-Nikodym derivative of h given $\mathscr{M}$.

1. Introduction. H. D. Brunk defined conditional expectation given a σ-lattice and established several of its properties in [1]. Subsequently S. Johansen [5] described a Radon-Nikodym derivative given a σ-lattice and showed that the Radon-Nikodym derivative was the conditional expectation in the appropriate case. Then H. D. Brunk and S. Johansen [2] proved an almost everywhere martingale convergence theorem for the Radon-Nikodym derivatives given an increasing sequence of σ-lattices. We shall establish norm convergence of these derivatives in L_1 and in the Orlicz spaces L^Φ, where Φ satisfies the Δ_2-condition. The theory of these Orlicz spaces can be found in [6], so we shall assume and build upon the results therein. Thereby, we can place fewer restrictions on Φ and obtain L_1-convergence as a byproduct.

2. Notation. Let $\mathscr{A}$ be a σ-algebra of subsets of a (non-empty) set Ω, and let μ be a non-negative (bounded) σ-additive function defined on $\mathscr{A}$.

For our purposes the following information about Φ will suffice: Φ is an even, convex function defined on the real numbers, R, with $\Phi(0) = 0$ and $\Phi(x) \neq 0$ for some x. Moreover, there exists $K > 0$ with $\Phi(2x) \leqq K\Phi(x)$ for all $x \in R$. This latter property is called the Δ_2-condition; it implies

$$(1)\quad \Phi(x+y) = \Phi\Big(2\Big(\frac{x+y}{2}\Big)\Big) \leqq K\Phi\Big(\frac{x+y}{2}\Big) \leqq \Big(\frac{K}{2}\Big)[\Phi(x) + \Phi(y)] .$$

Then L^Φ denotes the collection of (real valued) $\mathscr{A}$-measurable functions h defined on Ω with $\int_\Omega \Phi(h)d\mu < \infty$. Since Φ is convex and not