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STRONG CONCENTRATION OF THE SPECTRA OF SELF-ADJOINT OPERATORS

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Let H be a self-adjoint operator with spectral measure $E(S)$ over the Borel sets S of the real line. The spectrum of H is said to be strongly concentrated on S if whenever H_n converges strongly to H in the generalized sense it is true that $E_n(S)$ converges strongly to the identity. Sufficient conditions on H are given for this to occur for a given arbitrary Borel set S and necessary and sufficient conditions when S is the spectrum of H. In addition several more workable sufficient conditions are cited and a few examples illustrating the results are given.

Many authors have studied the changes in the spectra of a sequence of self-adjoint operators H_n as it converges strongly in some sense to a self-adjoint operator—e.g., [2], [3], [5], [6], [7, pp. 471-477], [8], [11]. It is known that while as point sets the spectra of H_n do not necessarily converge to the spectrum of H, nevertheless in some sense the spectra of H_n are concentrated on that of H. This spectral concentration phenomenon is described through the spectral measures E_n, E of the operators involved. In particular since $E(\Sigma)$ is the identity when Σ is the spectrum of H it is reasonable to say that the spectrum of the sequence H_n is concentrated on Σ if $E_n(\Sigma)$ converges to the identity as $n \to \infty$. Our main results concern necessary and sufficient conditions for this to occur for an arbitrary sequence H_n converging strongly to a fixed operator H. We make extensive use of the properties of the spectral measure $E(S)$ over the Borel sets S of the real line for which a general reference is [4] §§ X. 2 and XII. 2.

1. Preliminaries. Throughout this paper the following notation will be adhered to. H will denote a self-adjoint operator over a Hilbert space $\boldsymbol{H}$. Its domain will be denoted by $D(H)$ and its spectrum by Σ (which is always a closed subset of the real line R). The resolution of the identity of H will be denoted by $E(\lambda)$, $-\infty < \lambda < \infty$, and the associated projection-valued spectral measure by $E(S)$ over all Borel subsets S of R. By convention we take $E(\lambda)$ to be right continuous, i.e., $E(\lambda + 0) = E(\lambda)$. For a sequence of self-adjoint operators H_n, $n = 1, 2, \cdots$, over $\boldsymbol{H}$ the quantities $D(H_n)$, Σ_n, $E_n(\lambda)$, and $E_n(S)$ are defined accordingly.

According to a definition of Rellich (cf. [9] or [7, p. 429]) we