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HOMOLOGY OF A GROUP EXTENSION

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A topological method has been used by Ganea to derive the homology exact sequence of a central extension. In the same spirit a homology exact sequence is constructed for a group extension with certain homological restrictions. An immediate consequence is an exact sequence of Kervaire which is of some significance in algebraic K-theory.

Let

$$1 \longrightarrow N \rightarrow G \longrightarrow Q \longrightarrow 1 \tag{1}$$

be an extension of groups. Each element g of G induces an automorphism $\theta(g)\colon N \rightarrow N$ via $\theta(g)n = gng^{-1}$ for $n \in N$. In what follows we denote by $H_k(G)$ the kth homology group of G with coefficients in the additive group of integers Z, on which G operates trivially. Let Γ_k denote the subgroup of $H_k(N)$ generated by $\theta(g)_*c - c$, $c \in H_k(N)$, $g \in G$. We say that G operates trivially on $H_k(N)$ if $\Gamma_k = \{0\}$. Let $N \tilde{\times} G$ be the semi-direct product of N and G with respect to the operation $\theta(g)$ and let P_k denote the kernel of $\pi_*\colon H_k(N \tilde{\times} G) \rightarrow H_k(G)$, where $\pi\colon N \tilde{\times} G \rightarrow G$ is given by $\pi(n, g) = g$. We shall prove

THEOREM 1. *Suppose $n = 1$ or $H_k(N) = 0$ for $1 \leqq k \leqq n-1$ $(n \geqq 2)$. Then there exists an exact sequence*

$$\begin{aligned} P_{2n} \longrightarrow H_{2n}(G) \longrightarrow H_{2n}(Q) \longrightarrow P_{2n-1} \longrightarrow \cdots \longrightarrow P_{n+1} \longrightarrow H_{n+1}(G) \\ \longrightarrow H_{n+1}(Q) \longrightarrow H_n(N)/\Gamma_n \longrightarrow H_n(G) \longrightarrow H_n(Q) \longrightarrow 0\,. \end{aligned}$$

Further assume G operates trivially on $H_n(N)$ and that $H_1(Q) = 0$. Then there exists an exact sequence

$$H_{n+1}(N) \longrightarrow H_{n+1}(G) \longrightarrow H_{n+1}(Q) \longrightarrow H_n(N) \longrightarrow H_n(G) \longrightarrow H_n(Q) \longrightarrow 0\,.$$

We note that the first part of Theorem 1 for $n = 1$ is just Theorem 3.1 of [7].

Now we call an epimorphism $f\colon H \rightarrow H'$ *central* if Ker f is contained in the center of H. Let

$$\begin{tikzcd} & \tilde{N} \arrow[r] \arrow[d] & \tilde{G} \arrow[r] \arrow[d] & \tilde{Q} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 1 \\ & 1 & 1 & 1 & \end{tikzcd} \tag{2}$$