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SUPERSIMPLE SETS AND THE PROBLEM OF EXTENDING A RETRACING FUNCTION

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An infinite set A of natural numbers is called regressive if there is a (non-repeating) sequential ordering $a_0, a_1, \cdots$ of A and a partial recursive function f such that $A \subseteqq \text{domain}(f)$, $f(a_0) = a_0$, and $(\forall n)[f(a_{n+1}) = a_n]$; in case $a_0, a_1, \cdots$ is the natural ordering of A by increasing size of elements, A is called retraceable and f is said to retrace A. It sometimes happens that a retraceable set does not admit an everywhere-defined retracing function, or that it does not admit a finite-to-one retracing function. Question: does there exist an infinite set A of natural numbers such that A is retraced both by a total recursive function and by a finite-to-one partial recursive function, but not by a function which is both total recursive and finite-to-one? Via some theorems relating to D. A. Martin's notions of supersimple and superimmune sets of natural numbers, a strongly affirmative result is obtained.

1. Introduction. Let N denote the set of all natural numbers. In [3], Martin has termed *supersimple* any co-infinite Σ_1^0 subset S of N for which there does not exist a two-place total recursive function $f(x, y)$ with the properties: (i) $f(x, y)$ is characteristic (i.e., it maps N into $\{0, 1\}$); (ii) for each pair of distinct natural numbers x_1 and x_2, the sets $\{y \mid f(x_1, y\} = 0\}$ and $\{y \mid f(x_2, y) = 0\}$ are finite and disjoint; and (iii) for every x, the set $(N - S) \cap \{y \mid f(x, y) = 0\}$ is nonempty. Martin further suggests ([3, p. 306, footnote 2]) that a (not necessarily co-r.e.) set $I \subset N$ be called *superimmune* provided that I is infinite and that there is no two-place total recursive characteristic function $f(x, y)$ such that the sets $\{y \mid f(x, y) = 0\}$, $x = 0, 1, 2, \cdots$, are mutually disjoint (but not necessarily finite) and satisfy $\{y \mid f(x, y) = 0\} \cap I \neq \emptyset$ for all x. However, as we shall see, the adoption of that terminology would require the admission of supersimple sets having non-superimmune complements, which would not be in keeping with traditional recursion-theoretic nomenclature. Therefore we shall designate as *strongly superimmune* those sets which Martin recommended calling superimmune; and we shall say that an infinite set I is *superimmune* if its complement S (not necessarily a Σ_1^0 set) admits no total recursive function $f(x, y)$ satisfying (i), (ii) and (iii) above. If S is Σ_0^1 and has a strongly superimmune complement, we shall say that S is *strongly supersimple*.

The following theorem, due originally to Martin, is proved in [4]: