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A GENERALIZATION OF INJECTIVITY

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In a category of modules the notions of ρ-injectivity (with respect to a torsion radical ρ) and quasi-injectivity can be generalized to a notion of injectivity with respect to two preradicals simultaneously. Using this general definition an analog of Baer's condition for injectivity is obtained, as well as other generalizations of results for injective and quasi-injective modules. An alternate approach (not requiring the existence of injective envelopes) is given for abelian categories, with the results stated in dual form for projectivity.

In the first section of the paper we give some preliminary definitions and results, including a definition of density with respect to a preradical which is weaker than the standard one, and the definitions of preradicals rad^M and Rad^M associated with a module M (the smallest preradical and smallest torsion preradical, respectively, for which M is torsion). In the second section we define and study the notion of (ρ, σ)-injectivity, for preradicals ρ and σ. A module Q is called (ρ, σ)-injective if every homomorphism $f: N_0 \rightarrow Q$, where N_0 is a ρ-dense submodule of N and $\ker(f)$ is σ-dense in N, can be extended to N. This definition is motivated by a theorem of L. Fuchs [3, Lemma 1] giving a characterization of quasi-injectivity. Many of the results are motivated by those of G. Azumaya in his paper on M-projective and M-injective modules [1]. We prove that a module is M-injective if and only if it is (ρ, σ)-injective, where ρ is the identity functor and σ is either rad^M or Rad^M. This approach depends heavily on the existence of injective envelopes in categories of modules. In the third section of the paper we drop this assumption and obtain slightly weaker results valid in any abelian category. These results are stated in their dual form, for projectivity, and we show that our definition specializes, for modules with a projective cover, to that of M-projectivity.

1. Preliminary definitions and results. We will use the terminology of J.-M. Maranda [6]. A subfunctor ρ of the identity functor on an abelian category $\underline{A}$ is called a preradical of $\underline{A}$. Thus a preradical ρ of $\underline{A}$ assigns to each object A of $\underline{A}$ a subobject $\rho(A)$ and to each morphism $f: A \rightarrow B$ in $\underline{A}$ its restriction $\rho(f): \rho(A) \rightarrow \rho(B)$. It is said to be idempotent if $\rho^2 = \rho$ and is called a torsion preradical