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TWO THEOREMS OF GAUSS AND ALLIED IDENTITIES PROVED ARITHMETICALLY

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The product formulae of Gauss for the theta functions $\theta_4(0, q)$ and $(1/2)(-q)^{-1/8}\theta_2(0, (-q)^{1/2})$ have been derived in many ways by analytic means. In this paper these formulae are derived by enumerating certain types of partitions. The enumeration technique is shown to be applicable to more general results, and several important theorems in basic hypergeometric series are proved from suitable enumerations of partitions.

In [7], F. Franklin gave his now famous arithmetic proof of Euler's identity:

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(1/2)n(3n-1)} . \tag{1.1}$$

As is well-known, identity (1.1) can be deduced from Jacobi's triple product identity [10; p. 284]:

$$\prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n . \tag{1.2}$$

Several arithmetic proofs of (1.2) have appeared (e.g. [14; pp. 34-36], [15; pp. 10-12], [13], [16], and [2; pp. 561-562]). The known arithemetic proofs of (1.2) differ considerably from Franklin's proof of (1.1) probably because of the much greater generality of Jacobi's identity.

In this paper, we begin by considering the following two identities due to Gauss [8; p. 447, eqs. (14) and (10)]:

$$\begin{aligned} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 + q^{2n-1})} &= 1 + \sum_{n=1}^{\infty} (-1)^n q^{n(2n-1)}(1 + q^{2n}) \\ &= \frac{1}{2} i(-q)^{-1/8}\theta_2(0, (-q)^{1/2}) , \end{aligned} \tag{1.3}$$

and

$$\prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 + q^n)} = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} = \theta_4(0, q) . \tag{1.4}$$

These identities may be directly deduced from Jacobi's identity if one employs also the further result of Euler [10; p. 277]:

$$\prod_{n=1}^{\infty} (1 + q^n) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})} . \tag{1.5}$$