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RELLICH DENSITIES AND AN APPLICATION TO UNCONDITIONALLY NONOSCILLATORY ELLIPTIC EQUATIONS

JOHN PIEPENBRINK

Sufficient conditions for embeddings between weighted Sobolev spaces to be compact are derived. These theorems are generalizations of the well known selection principle of Rellich. These results are then applied to the study of the oscillational properties of self-adjoint second order elliptic equations. In addition to reproving some results of Headley and Swanson, new nonoscillation criteria are furnished for these equations.

1. Introduction. Let Ω be a domain, bounded or unbounded, in Euclidean n-space E^n, $p(x)$ a positive measurable function, $x = (x_1, \cdots, x_n)$, and $a(x)$ a symmetric matrix with measurable entries such that the smallest eigenvalue of $a(x)$ for each x in Ω positive. Define the weighted strong Sobolev spaces $H_\Omega(p)$ and $H_\Omega(p, a)$ as the closure of the sets of functions u, C^1 on Ω for which the integrals

$$\int_\Omega p(x)\,[u(x)]^2\,dx \tag{1.1}$$

$$\int_\Omega \{p(x)\,[u(x)]^2 + \sum a_{ij}(x)u_i(x)u_j(x)\}\,dx \tag{1.2}$$

are finite. The closures are taken with respect to norms given by (1.1) and (1.2). The weighted weak Sobolev spaces $W_\Omega(p)$ and $W_\Omega(p, a)$ consist of functions u with (1.1) or (1.2) respectively being finite. Here $u_i(x)$ is the distributional derivative $\partial u/\partial x_i$.

We will say that the pair (p, a) has the *strong Rellich compactness property* if the inclusion map $H_\Omega(p, a) \rightarrow H_\Omega(p)$ is compact. This means that each sequence in $H_\Omega(p, a)$ which is uniformly bounded in its norm has a subsequence which is convergent in the norm for $H_\Omega(p)$. The classical Rellich selection principle states that if Ω is bounded and smooth, then $(1, I)$ has the strong Rellich compactness property where I is the identity matrix. The *weak Rellich compactness property* is defined analogously with $W_\Omega(p)$, $W_\Omega(p, a)$ taking the place of $H_\Omega(p)$, $H_\Omega(p, a)$.

This paper investigates the case where $\Omega = E^n$, $n \geqq 2$. The arguments however apply eqully well to quasi-conical domains, i.e. domains which contain a cone $\{x \mid x \cdot v \geqq \alpha \mid x \mid\}$, where v is some unit vector, and α is a positive constant. Theorem 3.1 and 3.2 of § 3 provide sufficient conditions for (p, a) to have either the strong or