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A REMARK ON TONELLI'S THEOREM ON INTEGRATION IN PRODUCT SPACES

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This paper is concerned to show a connection between the validity of Tonelli's theorem on integration in the product of two measure spaces and the semifiniteness of the product measure. The classical Tonelli theorem is usually stated in a sigma-finite setting. It is shown in this paper, among other things, that in a product measure space, where one of the measures is sigma-finite and other one semifinite (not necessarily sigma-finite), Tonelli's theorem is valid only if the product measure is semifinite and on the other hand, if the product of any two measures is semifinite, then Tonelli's theorem is valid.

1. Let (X, U, β_1) and (Y, V, β_2) be any two arbitrary measure spaces where U and V are sigma-algebras of subsets of X and Y, respectively, and β_1 and β_2 are two nonnegative measures on U and V respectively. Let $U \times V$ be the smallest sigma-algebra containing all the measurable rectangles of $X \times Y$. The product measure $\beta_1 \times \beta_2$ (we call it β^*, for simplicity) is the restriction to $U \times V$ of the outer measure induced by the measure β on the algebra W consisting of the measurable rectangles of $X \times Y$ and their finite disjoint unions where for every measurable rectangle $P \times Q$, $\beta(P \times Q) = \beta_1(P)\beta_2(Q)$. (See [4], p. 254). β_1 is called semifinite if given A in U with $\beta_1(A) = \infty$, we can find B in U, $B \subset A$ and $0 < \beta(B) < \infty$. This definition, which at first glance seems to be less restricted than semifiniteness as defined in [4], p. 220, is actually equivalent to Royden's definition, as Lemma 1 in the next section shows. Every sigma-finite measure is semifinite, but not conversely. (For example, consider any non-sigma-finite regular Borel measure on a locally compact space or a counting measure on an uncountable set). The product measure $\beta_1 \times \beta_2$ may not be semifinite even when β_1 is sigma-finite and β_2 semifinite, as Example 1 shows. For the purpose of reference, let us state the following two well-known Theorems in a form, which is slightly different from that given in [2] or [4].

FUBINI'S THEOREM. *Let $f(x, y)$ be β^*-integrable on $U \times V$. Then both the iterated integrals of f are well-defined and*

$$\int f d\beta^* = \iint f d\beta_1 d\beta_2 = \iint f d\beta_2 d\beta_1 .$$