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REPRESENTATION THEORY OF ALMOST CONNECTED GROUPS

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Let G be a locally compact group and G_0 its connected component of the identity. If G/G_0 is compact, then G is a projective limit of Lie groups. In fact, there exist arbitrarily small compact normal subgroups $H \subseteq G$ such that G/H is a Lie group. Suppose H is such a compact, co-Lie subgroup of G. Then any unitary representation of G/H can be lifted to G in a natural way. Conversely, given a unitary representation π of G, one may ask whether it really lives on a Lie factor—that is, does there always exist a compact normal subgroup $H \subseteq G$ such that G/H is a Lie group and $\pi(h), h \in H$, is the identity operator? In this paper it is shown that this is indeed the case whenever π is irreducible (or more generally whenever π is factorial). The dual space $\hat{G}$ (=equivalence classes of irreducible unitary representations) is then realized as an inductive limit of the dual spaces of Lie groups. This inductive limit is first cast in a topological setting (using the dual topology on $\hat{G}$); and then, when G is also unimodular and type I, one obtains a measure-theoretic interpretation of the inductive limit (using the Plancherel measure). One application of these results is the fact that an almost connected group whose solvable radical is actually nilpotent must be a type I group.

1. Introduction. Let G be a locally compact group with left Haar measure dg. Denote by $\mathrm{Irr}(G)$ the collection of irreducible unitary representations of G, and by $\hat{G}$ the quotient space obtained from $\mathrm{Irr}(G)$ by the relation of unitary equivalence. When there is no possibility of confusion we shall fail to distinguish between a reperesentation $\pi \in \mathrm{Irr}(G)$ and its class $\{\pi\} \in \hat{G}$. It is possible to endow $\hat{G}$ with a locally compact (generally, non-Hausdorff) topology [6]; and with that topology $\hat{G}$ is called the *dual space* of G. Suppose in addition that G is unimodular and type I; then there is a unique positive Radon measure μ_G on $\hat{G}$ (called the Plancherel measure—see [3]) such that

$$(1.1) \qquad \int_G |f(g)|^2 dg = \int_{\hat{G}} \mathrm{Tr}\,[\pi(f)^*\pi(f)]d\mu_G(\pi)\ , \quad f \in L_1(G) \cap L_2(G)\ ,$$

where $\pi(f) = \int_G f(g)\pi(g)dg$.

Now by an almost connected group we mean a locally compact group G such that G/G_0 is compact, $G_0 =$ the connected component of the identity. It is well-known that such groups are projective limits