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AN ALGEBRA OF GENERALIZED FUNCTIONS ON AN OPEN INTERVAL: TWO-SIDED OPERATIONAL CALCULUS

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Let (a, b) be any open sub-interval of the real line, such that $-\infty \leqq a < 0 < b \leqq \infty$. Let $L^{\mathrm{loc}}(a, b)$ be the space of all the functions which are integrable on each interval (a', b') with $a < a' < b' < b$. There is a one-to-one linear transformation $\mathfrak{T}$ which maps $L^{\mathrm{loc}}(a, b)$ into a commutative algebra $\mathscr{A}$ of (linear) operators. This transformation $\mathfrak{T}$ maps convolution into operator-multiplication; therefore, this transformation $\mathfrak{T}$ is a useful substitute for the two-sided Laplace transformation; it can be used to solve problems that are not solvable by the distributional transformations (Fourier or bi-lateral Laplace).

In essence, the theme of this paper is a commutative algebra $\mathscr{A}$ of generalized functions on the interval (a, b); besides containing the function space $L^{\mathrm{loc}}(a, b)$, the algebra $\mathscr{A}$ contains every element of the distribution space $\mathscr{D}'(a, b)$ which is regular on the interval $(a, 0)$. The algebra $\mathscr{A}$ is the direct sum $\mathscr{A}_- \oplus \mathscr{A}_+$, where $\mathscr{A}_-$ (respectively, $\mathscr{A}_+$) $(a, 0)$ (respectively, to the interval $(0, b)$). There is a subspace $\mathscr{Y}$ of $\mathscr{A}$ such that, if $y \in \mathscr{Y}$, then y has an "initial value" $\langle y, 0- \rangle$ and a "derivative" $\partial_t y$ (which corresponds to the usual distributional derivative). If y is a function $f(\ )$ which is locally absolutely continuous on (a, b), then y belongs to $\mathscr{Y}$, the initial value $\langle y, 0- \rangle$ equals $f(0)$, and $\partial_t y$ corresponds to the usual derivative $f'(\ )$. If y is a distribution (such as the Dirac distribution) whose support is a locally finite subset of the interval (a, b), then both y and $\partial_t y$ belong to the subspace $\mathscr{Y}$. In case $a = -\infty$ and $b = \infty$, the subspace $\mathscr{Y}$ contains the distribution space $\mathscr{D}'_+$.

The resulting operational calculus takes into account the behavior of functions to the left of the origin (in case $a = -\infty$ and $b = \infty$, the whole real line is accounted for—whereas Mikusiński's operational calculus only accounts for the positive axis). Since the functions are not subjected to growth restrictions, the transformation $\mathfrak{T}$ is a useful substitute for the two-sided Laplace transformation (no strips of convergence need to be considered: see Examples 2.21 and the four problems 6.3–6.7). Problems such as

$$\frac{d^2}{dt^2} y + y = \sec \frac{\pi t}{2\alpha} \qquad (-\alpha < t < \alpha)$$