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INVARIANT FUNCTIONS OF AN ITERATIVE PROCESS FOR MAXIMIZATION OF A POLYNOMIAL

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Let P be a polynomial with real non-negative coefficients and variables $x_{i,j}$, $i = 1, \cdots, k$, $j = 1, \cdots, n_i$. Let $d = \sum_1^k n_i$. Let R_d be the d-dimensional real vector space. Let $\tilde{M}$ be the subset of R_d defined by

$$\tilde{M} = \left\{x \mid x \in R_d,\ x_{i,j} \geqq 0,\ \sum_{j=1}^{n_i} x_{i,j} = 1\right\}$$

where the symbols $x_{i,j}$ denote the components of x. If x is a vector in the interior of $\tilde{M}$, define $\tau(x)$ as the vector in $\tilde{M}$ with components $x'_{i,j}$ given by

$$x'_{i,j} = \frac{x_{i,j}\dfrac{\partial P}{\partial x_{i,j}}}{\sum_{h=1}^{n_i} x_{i,h}\dfrac{\partial P}{\partial x_{i,h}}}\,.$$

The expression on the right is evaluated at x. The transformation τ is defined on the boundary of $\tilde{M}$ by the same formula if the denominators do not vanish.

Let $\tilde{F}$ be the set of fixed points of τ in $\tilde{M}$. It is shown that if τ is a homeomorphism of $\tilde{M}$ onto itself, there is a set of $d - k$ functions $f_1, \cdots, f_{d-k}$ defined on $\tilde{M} - \tilde{F}$ such that $f_i(x) = f_i(\tau(x))$ for $x \in \tilde{M} - \tilde{F}$. The functions f_i are continuous and independent on an open dense subset of $\tilde{M} - \tilde{F}$. Explicit expressions for certain invariant functions are also obtained.

1. The transformation τ. The transformation τ defined in the introduction can be used to iteratively find local maxima for the polynomial P. It was shown by L. E. Baum and J. A. Eagon [1] that if P is a homogeneous polynomial with positive coefficients and if x is an element of $\tilde{M}$ such that $\tau(x)$ is defined then either $\tau(x) = (x)$ or $P(\tau(x)) > P(x)$. This result was generalized at the suggestion of 0. Rothaus by L. E. Baum and G. R. Sell [2] to arbitrary polynomials with positive coefficients.

It will be assumed in this paper that the transformation τ is a homeomorphism of $\tilde{M}$ onto itself. According to an unpublished result of L. E. Baum, τ is a homeomorphism of $\tilde{M}$ onto itself if and only if the expression for P as a sum of distinct monomials with positive coefficients contains monomials $c_{i,j}x_{i,j}{}^{w_{i,j}}$ for all $i=1, \cdots, k$, $j=1, \cdots, n_i$ where $c_{i,j} > 0$ and $w_{i,j}$ is an integer greater than zero. Since this condition is satisfied if and only if τ is defined on all of $\tilde{M}$, a necessary and sufficient condition that τ is a homeomorphism of $\tilde{M}$ onto itself