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ON THE BEHAVIOR OF PINCHERLE BASIS FUNCTIONS

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A basis $\{\alpha_n\}$ in the space of analytic functions on a disc $\{z: |z| < R\}$ is called a Pincherle basis if, for each $n(=0, 1, \cdots)$, the Taylor expansion of $\alpha_n(z)$ has z^n as its first nonvanishing term. The object of the present work is to examine such sequences to determine how behavior of the individual functions α_n is related to the property that $\{\alpha_n\}$ is a basis. Of particular interest are the zeros of the functions $\psi_n(z) = \alpha_n(z)/z^n$, and the case when each ψ_n is a linear function vanishing at a corresponding point z_n is studied in detail. There exist bases in which infinitely many of the z_n coincide with some point in the disc, or in which the z_n cluster at the origin. Nevertheless, the basis property can be correlated with various growth-rate conditions on $\{z_n\}$. For example, if the sequence $\{|z_0 z_1 \cdots z_{n-1}|^{1/n}\}$ converges to some number A, then the condition $A \geqq R$ is necessary and sufficient for $\{\alpha_n\}$ to be a basis. This and similar results are derived by using the automorphism theorem and properties of entire functions of exponential type. Correlations of this sort fail to materialize, however, for general (nonlinear) ψ_n, and certain phenomena encountered in this case are illustrated by examples involving nowhere vanishing ψ_n.

Although a great deal is now known about bases in topological linear spaces (see e.g. J. Marti [9]), the setting of analytic function spaces remains one of the most fruitful. In such spaces, primary interest attaches to the polynomial bases and the Pincherle bases. The latter are closely linked with the fundamental basis

$$\delta_n(z) = z^n \qquad (n = 0, 1, \cdots)\,, \tag{1.1}$$

which leads to considerable simplification, but Pincherle bases still turn out to be vastly more complicated than $\{\delta_n\}$. This will be imply evident in our discussion of the correlation between the individual functions α_n and the basis property. Certain aspects of the problem, discussed in [1] and [2], will be drawn on as needed. The automorphism theorem (about which more will be said presently) remains our principal tool and permits us to avoid use of the elaborate theory of basic series, developed for polynomials by J. M. Whittaker [11] and extended by W. F. Newns [10].

Let us recall a few of the relevant concepts. With only minor changes, the notation and terminology of [3] will be used throughout. Thus, we take the underlying space as the Fréchet space $\mathscr{F}_R$ of all functions analytic on a fixed open disc $N_R(0)$ of radius R $(0 < R \leqq +\infty)$