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DOUBLE COMMUTANTS OF WEIGHTED SHIFTS

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Let A be an operator (bounded linear transformation) on a (complex) Hilbert space. If the double commutant of A is equal to the weakly closed algebra (with identity) generated by A we shall say that A belongs to the class (dc).

By the von Neumann Double Commutant Theorem any Hermitian operator is in (dc), and it is well known that any operator on a finite dimensional space is in (dc). A generalization of this latter fact was proven in [10]. The object of this paper is to show that any one-sided weighted shift is in (dc) and that a two-sided weighted shift is in (dc) if and only if it is not invertible.

In their paper "The commutants of certain Hilbert space operators", [9], Shields and Wallen show that any one-sided shift *with nonzero weights* generates a maximal abelian weakly closed algebra and therefore is à fortiori in (dc). Related work on commutants of weighted shifts has also been done by Gellar. (See [2], [3], and [4].)

1. Definitions and notation.

If $\mathfrak{H}$ is a Hilbert space we denote be $B(\mathfrak{H})$ the algebra of all operators on $\mathfrak{H}$. For A belonging to $B(\mathfrak{H})$ we denote by $\mathfrak{A}_A$ the weakly closed algebra with identity generated by A, by $\mathfrak{A}'_A$ the commutant of A, and by $\mathfrak{A}''_A$ the double commutant of A. (The reader is referred to [1, p. 1] for definitions of commutant and double commutant.) The class (dc) is the class of all operators A on Hilbert space such that $\mathfrak{A}_A = \mathfrak{A}''_A$.

Let $\mathfrak{H}$ be a separable Hilbert space, let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathfrak{H}$, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a bounded sequence of scalars. Define S on $\mathfrak{H}$ by $Se_n = \alpha_{n+1}e_{n+1}$ for all n, extending linearly and continuously. Then S is called a *one-sided weighted shift* on $\mathfrak{H}$.

Now let $\{e_n\}_{n=-\infty}^{\infty}$ be a two-side orthonormal basis for $\mathfrak{H}$ and let $\{\alpha_n\}_{n=-\infty}^{\infty}$ be a doubly infinite bounded sequence of scalars. Definite S on $\mathfrak{H}$ by $Se_n = \alpha_{n+1}e_{n+1}$ for all n, extending linearly and continuously as before. Then S is called a *two-sided* weighted shift on $\mathfrak{H}$.

Finally let $\mathfrak{H}$ be a finite dimensional Hilbert space and let $\{e_0, \cdots, e_k\}$ be an orthonormal basis for $\mathfrak{H}$. Let $\alpha_1 \cdots \alpha_k$ be scalars. Define S on $\mathfrak{H}$ by $Se_n = \alpha_{n+1}e_{n+1}$ for $0 \leqq n < k$, $Se_k = 0$, extending linearly. Then S is called a *finite* weighted shift on $\mathfrak{H}$.

In the forthcoming we shall drop the term "weighted" and speak simply of *shifts*.

A *backward* (one-sided, two-sided, or finite) shift is the adjoint of a (one-sided, two-sided, or finite) shift respectively. Note that