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COMPLEMENTATION PROBLEMS FOR THE BAIRE CLASSES

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This paper initiates a study of the classes of Baire measurable functions on the unit interval I from the standpoint of the theory of spaces of continuous functions. For each countable ordinal α, the αth Baire class $\mathfrak{B}_\alpha$ has a representation as $C(\Omega_\alpha)$, where Ω_α is a certain compactification of the discrete set I. For $1 \leqq \alpha < \beta$, $\mathfrak{B}_\alpha$ is a closed subalgebra of $\mathfrak{B}_\beta$. The principal result proved here is the fact that $\mathfrak{B}_\alpha$ is always uncomplemented as a closed subspace of $\mathfrak{B}_\beta$. The method of proof relies on a detailed analysis on the canonical onto map $\phi: \Omega_\beta \rightarrow \Omega_\alpha$ induced by the imbedding of $\mathfrak{B}_\alpha$ in $\mathfrak{B}_\beta$, and consists of showing that this map admits no "averaging operator." It depends heavily on recent results in the theory of averaging operators due to S. Z. Ditor.

In this paper scalars and functions are real valued. However, the arguments extend easily to the complex case. In the last section we show how corresponding results may be obtained when I is replaced by any uncountable compact metric space.

1. The Baire classes as function algebras.

We shall start by recalling classical definitions and facts concerning the Baire classes of functions on the unit interval I. Let $C(I)$ be the class of all real continuous functions on I with supremum norm. Denote by $\mathfrak{B}_1$ the class of all bounded functions which are pointwise limits of sequences of functions in $C(I)$, and for each countable ordinal α inductively define $\mathfrak{B}_\alpha$ to be the class of all bounded functions on I which are pointwise limits of sequences of functions in $\bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. We call $\mathfrak{B}_\alpha$ the class of Baire functions of order α.

There is another approach to $\mathfrak{B}_\alpha$. Each countable ordinal α is even or odd as follows: 1 is odd and each limit ordinal is even; the immediate successor of an even ordinal is odd, and of an odd ordinal is even. Let F_0 be the class of all closed subsets of I; and F_1 be the class of countable unions of sets in F_0. For each α

(i) F_α is the class of all countable unions of sets in $\bigcup_{\beta<\alpha} F_\beta$, if α is odd;

(ii) F_α is the class of all countable intersections of sets in $\bigcup_{\beta<\alpha} F_\beta$, if α is even.

Correspondingly, let G_0 be the class of all open subsets of I, and G_1 be all countable intersections of G_0 sets. For each α

(iii) G_α is the class of all countable intersections of sets in $\bigcup_{\beta<\alpha} G_\beta$,