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THE CONSTRUCTIVE THEORY OF COUNTABLE ABELIAN P-GROUPS

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The purpose of this paper is to develop the theory of abelian p-groups along constructive lines. To this end a constructive theory of ordinal numbers and an axiomatic treatment of the notion of height are presented. The classical theorems of Zippin and Ulm concerning existence and uniqueness of countable p-groups with prescribed invariants are proved in a finitistic setting.

1. Introduction. Throughout this paper the letter p will denote a fixed prime, and we shall use the word *group* to mean an abelian p-group. The idea of a p-group illustrates how the constructive point of view directs our attention to the manner in which a mathematical object is presented, and not merely to its absolute structure. When we say that an element x of a group has order a power of p, we mean that we *can find* an integer n such that $p^n x = 0$. The interpretation of this is that we possess an algorithm that will produce, in a finite number of steps (bounded in advance), such an integer n.

Decision problems play an important role in this approach. It is easy to come up with an algorithm that produces a sequence of 0's and 1's so that no one knows whether it will ever produce a 1. The nth output of one such algorithm is 1 if and only if 100 consecutive 7's appear in the first n digits of the decimal expansion of π. However, the idea of a decision problem does not depend upon the existence of such algorithms. It is a question of what information is at your disposal; we could just as well imagine that we were examining the output of an algorithm, possibly a completely transparent one, of whose nature we were ignorant. One such problem is deciding whether two elements of a set are equal or not. A set is called *discrete* if we can settle this question for any two of its elements. The set of sequences of 0's and 1's is not discrete, for we cannot necessarily tell whether a given sequence is equal to a sequence of all 0's.

Countability, in the constructive setting, carries a different connotation from the classical one of "not too big." Here the significance is that we can call for the elements one by one, and given an element we can compute its place in that sequence. A subset of a countable set need not be countable. The set of exponents n for which Fermat's conjecture is true is such a subset. The dif-