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RESTRICTING ISOTOPIES OF SPHERES

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In this note we consider the problem of determining whether isotopic homeomorphisms of S^n that agree on a subset X of S^n are isotopic by an isotopy that is fixed on X. In particular, in the PL category, an affirmative answer is obtained for X a locally unknotted closed cell or an unknotted sphere.

If X and Y are polyhedra and h_0 and h_1 are homeomorphisms of X onto Y, then an *isotopy* between h_0 and h_1 is a homeomorphism $H: X \times I \rightarrow Y \times I (I = [0, 1])$ such that $H(x, t) = (h_t(x), t)$ for all $(x, t) \in X \times I$. Two embeddings f, g of X in Y are said to be *ambient isotopic* if there is an isotopy $H: Y \times I \rightarrow Y \times I$ such that $h_0 = \text{id.}$, and $h_1 f = g$. The isotopy H is *fixed on* $A \subset Y$ if $H(x, t) = (x, t)$ for all $(x, t) \in A \times I$. Let S^n denote the standard n-sphere, E^n Euclidean n-space, Δ^k a k-simplex in some combinatorial triangulation of S^n or E^n, and let "PL" denote "piecewise linear." If $k < n$ we regard S^n as the $(n - k)$-fold suspension of S^k, so there is a natural inclusion $S^k \subset S^n$. A PL embedding $i: S^k \rightarrow S^n$ is *unknotted* if $(S^n, i(S^k)) \underset{\approx}{PL} (S^n, S^k)$, which is always the case if $k \leqq n - 3$. Clearly an unknotted sphere Σ^k in S^n is PL locally flat; i.e., for each point $x \in \Sigma^k$, there is a neighborhood U of x in S^n such that

$$(U, U \cap \Sigma^k) \underset{\approx}{PL} (E^n, E^k) .$$

The main results of this paper are the following:

THEOREM 1. *Let $X = \Delta^k$ or $X = S^k$, and let $i: X \rightarrow S^n$ be a PL-embedding, unknotted if $X = S^k$, locally unknotted if $X = \Delta^k$. If f and g are PL-homeomorphisms of S^n that are ambient isotopic, and if $f \mid i(X) = g \mid i(X)$, then f and g are PL ambient isotopic fixing $i(X)$.*

THEOREM 2. *Let $\Sigma^k \subset S^n$ be unknotted, $n \geqq 5$, $k \neq 3$, and f and g be homeomorphisms S^n that are isotopic and agree on Σ. Then f and g are ambient isotopic fixing Σ.*

If $k \leqq n - 3$, then Theorem 1 is a special case of [2]. Note that in Theorem 2, we do not require f and g to be PL.

The key step in the proof of these theorems is

LEMMA 3. *Let X be a k-simplex in S^n or the standard k-sphere $S^k \subset S^n$. If f is an orientation preserving PL-homeomorphism of S^n*