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A DECOMPOSITION FOR $B(X)^*$ AND UNIQUE HAHN-BANACH EXTENSIONS

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For a Banach space X, let $B(X)$ be the space of all bounded linear operators on X, and $\mathscr{C}$ the space of all compact linear operators on X. In general, the norm-preserving extension of a linear functional in the Hahn-Banach theorem is highly non-unique. The principal result of this paper is that, for $X = c_0$ or l^p with $1 < p < \infty$, each bounded linear functional on $\mathscr{C}$ has a unique norm-preserving to $B(X)$. This is proved by using a decomposition theorem for $B(X)^*$, which takes on a special form for $X = c_0$ or l^p with $1 < p < \infty$.

1. DEFINITION 1.1. A basis $\{e_i\}$ for a Banach space X having coefficient functionals e_i^* in X^* is called unconditional if, for each x, $\sum_{i=1}^{\infty} e_i^*(x)e_i$ converges unconditionally. The basis is called monotone if $||U_m x|| < ||x||$ for all $x \in X$ and positive integers m, where $U_m x = \sum_{i=1}^{m} e_i^*(x)e_i$.

PROPOSITION 1.2. *If X has a monotone, unconditional basis $\{e_i\}$, then $B(X)^* = \mathscr{C}^* + \mathscr{C}^{\perp}$, where $\mathscr{C}^*$ is a subspace of $B(X)^*$ isomorphically isometric to the space of bounded linear functionals on $\mathscr{C}$, and $\mathscr{C}^{\perp}$ annihilates $\mathscr{C}$. Furthermore, the associated projection from $B(X)^*$ onto $\mathscr{C}^*$ has unit norm.*

Proof. If $T \in B(X)$, then $T(x) = \sum_{i=1}^{\infty} f_i^T(x)e_i$ for each $x \in X$, where $f_i^T \in X^*$. For each T and i, let T_i be defined by $T_i(x) = f_i^T(x)e_i$ for all x. Also, for each $F \in B(X)^*$, define $G \in B(X)^*$ by $G(T) = \sum_{i=1}^{\infty} F(T_i)$. Note that this sum converges. Otherwise, we have $\sum_{i=1}^{\infty} |F(T_i)| = \lim_{n\to\infty} F[\sum_{i=1}^{n} SgF(T_i)\cdot T_i] = +\infty$, and then

$$\lim_{n\to\infty} ||\textstyle\sum_{i=1}^{n} SgF(T_i)\cdot T_i|| = \infty .$$

Then by using an absolutely convergent series, it is easy to construct an element $y \in X$: $\lim_{n\to\infty} ||\sum_{i=1}^{n} SgF(T_i)\cdot T_i(y)|| = \infty$. Therefore, $\sum_{i=1}^{\infty} f_i^T(y)e_i$ converges while $\sum_{i=1}^{\infty} SgF(T_i)\cdot f_i^T(y)e_i$ does not, which contradicts the fact that an unconditionally convergent series is bounded multiplier convergent. See [3], p. 19.

Note that the norm of G restricted to $\mathscr{C}$ is equal to the norm of G on $B(X)$, since by monotonicity $||\sum_{i=1}^{n} T_i|| \leqq ||T||$ for each n and $T \in B(X)$. Also, F and G agree on $\mathscr{C}$, because $\mathscr{C}$ is the closure of the set of all T for which only a finite number of the f_i^T are nonzero. Hence the projection defined by $PF = G$ has unit norm, since