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THE CLASS OF RECURSIVELY ENUMERABLE SUBSETS OF A RECURSIVELY ENUMERABLE SET

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For any set α, let θA^{α} denote the index set of the class of all recursively enumerable (r.e.) subsets of α (i.e., if $\{W_x\}_{x \geqq 0}$ is a standard enumeration of all r.e. sets, $\theta A^{\alpha} = \{x \mid W_x \subset \alpha\}$.) The purpose of this paper is to examine the possible Turing degrees of the sets θA^{α} when α is r.e. It is proved that if $\boldsymbol{b}$ is any nonrecursive r.e. degree, the Turing degrees of sets θA^{α} for α r.e., $\alpha \in \boldsymbol{b}$, are exactly the degrees $\boldsymbol{c} > \mathbf{0}'$ such that $\boldsymbol{c}$ is r.e. in $\boldsymbol{b}$.

Index sets of form θA^{α} appear to have useful properties in the study of the partial ordering of all index sets under one-to-one reducibility. For instance, in the case where α is a nonrecursive incomplete r.e. set, the index set $\overline{\theta A^{\alpha}}$ was used in [1] to provide an example of an index set which is neither r.e. nor productive. In [2] it is shown that if the Turing degree of α is not $\geqq \mathbf{0}'$, then the set θA^{α} is at the bottom of c discrete ω-sequences of index sets (i.e., linearly ordered chains of index sets such that no index sets are intermediate between the elements of the chain.) In particular, such a set θA^{α} has at least two nonisomorphic immediate successors in the partial ordering of index sets.

It is natural to ask: What relation, if any, exists between the Turing degree of α and that of θA^{α}? In the case where α is co-r.e., it is easy to see that neither degree determines the other, since $\overline{\theta A^{\alpha}}$ is r.e. and hence has degree $\mathbf{0}$ or $\mathbf{0}'$ (by Rice's theorem [5]), independently of the degree of α; while both $\mathbf{0}$ and $\mathbf{0}'$ contain sets θA^{α} for $\alpha \in \mathbf{0}$. In this paper it is shown that when α is r.e., the situation is similar, though more complicated. It was shown in [3, Theorem 1] that if β is a complete r.e. set, then θA^{β} is a complete Π_2^0 set. On the other hand, C. G. Jockush, Jr. has constructed an example (unpublished) of an effectively simple set γ such that θA^{γ} has degree $\mathbf{0}'$. Since β and γ both have degree $\mathbf{0}'$ [4], this shows that when α is r.e., the degree of α need not determine that of θA^{α}. The main result of this paper shows that these examples are extremal cases of the fact that when α is r.e., the degree of θA^{α} can take on all possible values within certain obvious restrictions. More precisely, we prove the following:

THEOREM. *Let $\boldsymbol{b}$ be a nonrecursive* r.e. *degree. Let*