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RADIAL QUASIHARMONIC FUNCTIONS

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A function s on a Riemannian manifold is called quasiharmonic if it satisfies $\Delta s = 1$, where Δ is the Laplace-Beltrami operator $d\delta + \delta d$. Existence of quasiharmonic functions with various boundedness properties has thus far been investigated by means of useful implicit tests. We now ask: Can such functions be formed by direct construction, in a manner accessible to computation if need be?

1. We shall present our approach to the problem in the setup of a Riemannian N-ball

$$B_\alpha = \{r < 1 \mid ds\} \tag{1}$$

endowed with the generalized Poincaré metric

$$ds = \lambda(r)\,|dx|,\ \lambda(r) = (1 - r^2)^\alpha,\ \alpha \in \boldsymbol{R}\ , \tag{2}$$

where $r = |x|$, $x = (x^1, \cdots, x^N)$. In [16] we proved that there exist bounded quasiharmonic functions on B_α if and only if $\alpha \in (-1, 1/(N-2))$. We shall now show that this in turn is necessary and sufficient for the boundedness of an explicitly constructed function $s(r)$, given in No. 3 below. *Thus the boundedness of this single function characterizes the existence of bounded quasiharmonic functions on* B_α.

We shall call, for brevity, a function radial if it depends on r only. A simple consequence of our result is that there exist bounded radial quasiharmonic functions if and only if there exist bounded quasiharmonic functions.

We expect that our approach is extendable to other classes of quasiharmonic and biharmonic functions as well, and to other Riemannian manifolds which are invariant under rotation. In particular, *there exist negative radial quasiharmonic functions on every* B_α.

2. The proof of our main result will be divided into Lemmas 1-6. We start by formulating the equation:

LEMMA 1. *A function $s(r)$ satisfies*

$$\Delta s = 1 \tag{3}$$

on B_α if and only if

$$s'' + \left(\frac{N-1}{r} - \frac{2(N-2)\alpha r}{1-r^2}\right)s' + (1-r^2)^{2\alpha} = 0\ . \tag{4}$$