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ON A PROBLEM OF COMPLETION IN BORNOLOGY

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In this note an example is given to show that the bornological completion of a polar space need not be polar. Also, a theorem of Grothendieck's type is proved, from which necessary and sufficient conditions for the completion of a polar space to be again polar are derived.

1. Notation and terminology are as in [4]. In particular, b.c.s. means a locally convex, bornological linear space over the scalar field of real or complex numbers.

In [4, 5. p. 160] Hogbe-Nlend lists, among unsolved problems in bornology, the following one, which was first raised by Buchwalter in his thesis [1, Remarque, p. 26]:

Is the bornological completion of a polar b.c.s. again polar?

The purpose of this note is to exhibit an example that answers this question in the negative. We also prove a theorem of Grothendieck's type for regular b.c.s. with weakly concordant norms, which enables us to give necessary and sufficient conditions for the completion of a polar b.c.s. to be polar.

2. For each n let the double sequence $a^n = (a^n_{ij})$ be defined by $a^n_{ij} = j$ for $i \leqq n$ and all j, $a^n_{ij} = 1$ for $i > n$ and all j, and denote by E_n the normed space of scalar-valued double sequences (x_{ij}) with only finitely many nonzero terms, under the norm

$$\|(x_{ij})\|_n = \sup_{i,j} \frac{|x_{ij}|}{a^n_{ij}} . \tag{1}$$

Let E be the bornological inductive limit of the spaces E_n; thus $E = E_n$ algebraically, and a set $B \subset E$ is bounded for the inductive limit bornology if and only if there exist positive integers n, k such that $\|(x_{ij})\|_n \leqq k$ for all $(x_{ij}) \in B$. It is easily seen that E is a polar b.c.s. whose dual $E^\times$ consists of all scalar-valued double sequences (u_{ij}) such that

$$\sum_{i,j=1}^{\infty} a^n_{ij} |u_{ij}| < \infty \qquad \text{for all } n .$$

By [1, Théorème (2.8.15)] the completion $\hat{E}$ of E is given by $\hat{E} = \varinjlim \hat{E}_n$ (bornological inductive limit), where $\hat{E}_n$ is the completion of the normed space E_n, i.e., the Banach space of scalar-valued double sequences (x_{ij}) such that $\lim_{i,j\to\infty} x_{ij}/a^n_{ij} = 0$ under the norm (1). It also