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ON THE SEMISIMPLICITY OF GROUP RINGS OF LINEAR GROUPS

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In this paper we study the semisimplicity problem for group rings of linear groups. We prove the linear group analog of a result which constitutes part of the solution of the semisimplicity problem for solvable groups. Since all of the necessary group ring lemmas have appeared elsewhere, the work here is strictly group theoretic. We consider the possibility of a linear group being covered by a finite union of root sets of centralizer subgroups.

Let $K[G]$ denote the group ring of G over the field K. Probably the most fascinating and difficult question asked about this ring is when is it semisimple, that is when does its Jacobson radical $JK[G]$ vanish. If K has characteristic 0, then by a result of Amitsur [1] $JK[G] = 0$ for all fields K which are not algebraic over the rationals and in all likelihood $K[G]$ is always semisimple. Thus the real interest is now in characteristic $p > 0$. At this time there is not even a reasonable conjecture as to the answer here and so it is necessary and important that a large number of special cases be studied. So far the only nontrivial family of groups for which this problem has been solved is in fact the solvable groups, that is the groups of finite derived length. It appears that the next family of interest will be the linear groups since some interesting work in this direction has already appeared in [4]. In this paper we study the semisimplicity problem for linear groups.

Let G be a group and let H be a subgroup. We say that H has locally finite index in G and write $[G: H] = \text{l.f.}$ if for all finitely generated subgroups S of G we have $[S: S \cap H] < \infty$. We say that G is a Δ-group if $G = \Delta(G)$, that is if all conjugacy classes of G are finite. The result on solvable groups was proved in a series of three papers [3], [2], and [5] and is as follows.

THEOREM. (*Hampton, Passman, and Zalesskii.*) *Let K be a field of characteristic $p > 0$ and let G be a solvable group. Then $JK[G] \neq 0$ if and only if there exists an element $x \in \ni(G)$ (a certain characteristic subgroup of G) of order p with $[G: C_G(x)] = l.\text{f}.$*

We remark that $\ni(G)$ is a particular normal Δ-subgroup of a solvable group which is defined in [5] and we propose to call it the Zalesskii subgroup of G.