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ENDOMORPHISM RINGS OF FINITELY GENERATED PROJECTIVE MODULES

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Over a ring A let P_A be a finitely generated projective right A-module with A-endomorphism ring B. Anderson has called P_A an injector, perfect injector, projector, perfect projector, if the functor $F = {}_BP \otimes_A (\ )$ preserves injectives, injective hulls, projectives, projective covers, respectively. Call P_A a flatjector if F preserves flat modules. Injectors, flatjectors, and projectors are characterized. The radical of a module over B is studied, and necessary and sufficient conditions are given for the radical of B to be left T-nilpotent. Perfect injectors are characterized. Previous characterizations of perfect projectors have assummed the ring A to be left perfect. Here characterizations are obtained using substantially weaker conditions on P_A.

Over a ring A let P_A be a finitely generated projective right A-module with A-endomorphism ring B. Let T be the trace of P_A and $P^* = \mathrm{Hom}_A(P, A)$ be the A-dual of P. Injectors, flatjectors, and projectors are characterized in §2, the characterizations of injectors and projectors being extensions of results due to Anderson [1]. For example, P_A is a flatjector if and only if ${}_BP$ (equivalently ${}_AP^* \otimes {}_BP$) is flat. P_A is a projector if and only if the natural map $\eta\colon {}_AP^* \otimes P \to {}_AT$ is a projective cover. It is shown that P_A is a flatjector if and only if ${}_AP^*$ is an injector. Furthermore, every projector is a flatjector, the two being equivalent for perfect rings. Examples are given of a flatjector that is not a projector and a nonperfect ring where every flatjector is a projector.

In §3 it is shown that the B-radical of ${}_BP \otimes {}_AX$ is isomorphic to ${}_BP \otimes {}_AJ(TX)$. In particular the B-radical of ${}_BP$ is isomorphic to ${}_BP \otimes {}_AJ(T)$. A definition of U-dominant codimension dual to the definition of V-dominant dimension given in [8] is introduced. The radical N of B being left T-nilpotent is characterized in terms of the full subcategories $\mathscr{C}_1({}_AP^*)$ of ${}_A\mathfrak{M}$ consisting of all left A-modules of ${}_AP^*$-dominant codimension $\geqq 1$, and $\mathfrak{D}_1(Q_A)$ of $\mathfrak{M}_A$ consisting of all right A-modules of Q_A-dominant dimension $\geqq 1$ (where $Q_A = \mathrm{Hom}_B(P, W)$ for ${}_BW$ an injective cogenerator). It is shown that N is left T-nilpotent if and only if JT is left T-nilpotent, or equivalently $J(\bigoplus_I P^*)$ is small in $\bigoplus_I P^*$ for any index set I.

In §4 perfect injectors are characterized in terms of their trace ideal and certain conditions on large submodules. Anderson [1] has studied perfect projectors when the ring A is left perfect. Here