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ON THE NUMBER OF POLYNOMIALS OF AN IDEMPOTENT ALGEBRA, II

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In part I of this paper a conjecture was formulated according to which, with a few obvious exceptions, the sequence $\langle p_n(\mathfrak{A})\rangle$ of an idempotent algebra is eventually strictly increasing. In this paper this conjecture is verified for idempotent algebras satisfying $p_2(\mathfrak{A}) = 0, p_3(\mathfrak{A}) > 0$, and $p_4(\mathfrak{A}) > 0$. In fact, somewhat more is proved:

THEOREM. **Let $\mathfrak{A}$ be an idempotent algebra with no essentially binary polynomial and with essentially ternary and quaternary polynomials. Then the sequence**

$$p_3(\mathfrak{A}), p_4(\mathfrak{A}), \cdots, p_n(\mathfrak{A}), \cdots$$

is strictly increasing, that is, for all $n \geqq 2$

$$p_n(\mathfrak{A}) + 1 \leqq p_{n+1}(\mathfrak{A}) .$$

The proof starts in §2 where a lemma of K. Urbanik is modified to show that the proof splits naturally into three cases. §§3 and 4 handle the first two cases. In §5 the third case is analyzed and it is proved that it splits into two further cases that are settled in §§6 and 7. In each of these sections examples are provided that the case under consideration is not void.

For the undefined concepts and basic results the reader is referred to [2].

Examples of algebras satisfying the conditions of the Theorem abound. On a two element Boolean algebra $\{0, 1\}$ the operation $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ defines such an algebra.

2. The classification. An algebra $\mathfrak{A} = \langle A; F\rangle$ is *idempotent* if every operation $f \in F$ has type (arity) > 0, and $f(a, \cdots, a) = a$ for all $a \in A$. All algebras considered in this paper are assumed to have more than one element. An n-ary polynomial p of $\mathfrak{A}$ (that is, an n-ary function or A composed from functions in F) *depends on* x_i $(1 \leqq i \leqq n)$ if there exist $a_1, \cdots, a_n, a'_i \in A$ with $p(a_1, \cdots, a_i, \cdots, a_n) \neq p(a_1, \cdots, a'_i, \cdots, a_n)$; p is *essentially n-ary*, if p depends on $x_1, \cdots, x_n$. For $n \geqq 2$, let $p_n(\mathfrak{A})$ denote the number of essentially n-ary polynomials.

In this paper we shall deal exclusively with idempotent algebras satisfying

$$p_2(\mathfrak{A}) = 0, p_3(\mathfrak{A}) \neq 0 , \quad \text{and} \quad p_4(\mathfrak{A}) \neq 0 .$$

The sequence $\langle p_n(\mathfrak{A})\rangle$ is strictly increasing because $\mathfrak{A}$ must have