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# THE LATTICE OF CLOSED IDEALS AND $a^*$-EXTENSIONS OF AN ABELIAN $l$-GROUP

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**An $l$-ideal $A$ of an $l$-group $G$ is *closed* if $x \in A$ whenever $x = \vee a_i, 0 \leqslant a_i \in A$. The intersection of any collection of closed $l$-ideals of $G$ is again a closed $l$-ideal of $G$. Hence the set $\mathscr{K}(G)$ of all closed $l$-ideals of $G$ is a complete lattice under inclusion. In the present paper this lattice is studied, as well as $l$-group extensions which preserve it. A common generalization of the essential closure of an archimedean $l$-group and the Hahn closure of a totally-ordered abelian group is obtained.**

*Unless otherwise specified all l-groups will be assumed abelian.* Set-theoretic union and intersection will be written $\cup$ and $\cap$, respectively. The lattice of all $l$-ideals of an $l$-group $G$ will be denoted $\mathscr{L}(G)$; the join operation in $\mathscr{L}(G)$ will be written $\vee$ (to be differentiated by context from the $l$-group operation). The join operation in $\mathscr{K}(G)$ will be written $\bigcup$. A subset $D$ of a partially ordered set $S$ will be called a *dual ideal* if $x \in D$ whenever $x \geqslant y$ for some $y \in D$.

$G(g)$ will denote the smallest $l$-ideal of $G$ containing $g \in G$. $\bar{A}$ will denote the smallest closed $l$-ideal of $G$ containing $A \in \mathscr{L}(G)$. We have $\bar{A}^+ = \{\bigvee a_i \mid 0 \leqslant a_i \in A\}$. ([5], Lemma 3.2).

$A \in \mathscr{L}(G)$ is a *regular* subgroup of $G$ if it is maximal in $\mathscr{L}(G)$ without some $g \in G$; in this case $A$ is also called a *value* of $g$. If $A$ is the only value of some $g \in G$, then $A$ is a *special* subgroup of $G$. Each special subgroup of $G$ is closed. ([4], Prop. 4.1). If each $g \in G$ has only finitely many values, then $G$ is *finite-valued.* An $l$-ideal of $G$ is *prime* if it is the intersection of a chain of regular subgroups of $G$. An $l$-ideal of $G$ which contains a closed prime subgroup of $G$ is itself a closed prime subgroup. ([5], Lemma 3.3).

We conclude the introduction by reviewing the important results in [10]. Let $\Lambda$ be a root system (i.e., $\Lambda$ is a partially ordered set and no two noncomparable elements of $\Lambda$ have a lower bound in $\Lambda$). Let $V(\Lambda, R)$ denote the group of all real-valued functions on $\Lambda$ whose support satisfies the *ACC*. $\lambda \in \Lambda$ is a *maximal component* of $v \in V(\Lambda, R)$ if $\lambda$ belongs to the support of $v$ but no element of $\Lambda$ exceeding $\lambda$ belongs to the support of $v$. Define $v > 0$ if and only if $v(\lambda) > 0$ for each maximal component $\lambda$ of $v$. Then $V(\Lambda, R)$ is an $l$-group. If $\lambda \in \Lambda$, then $V_\lambda = \{v \in V(\Lambda, R) \mid v(\alpha) = 0 \text{ for all } \alpha \geqslant \lambda\}$ is a closed regular subgroup of $V(\Lambda, R)$; moreover, these are the only closed regular subgroups of $V(\Lambda, R)$.