PACIFIC JOURNAL OF MATHEMATICS
Vol. 47, No. 2, 1973

A HAUSDORFF-YOUNG THEOREM FOR REARRANGEMENT-INVARIANT SPACES

COLIN BENNETT

The classical Hausdorff-Young theorem is extended to the setting of rearrangement-invariant spaces. More precisely, if $1 \leqq p \leqq 2$, $p^{-1} + q^{-1} = 1$, and if X is a rearrangement-invariant space on the circle T with indices equal to p^{-1}, it is shown that there is a rearrangement-invariant space $\hat{X}$ on the integers Z with indices equal to q^{-1} such that the Fourier transform is a bounded linear operator from X into $\hat{X}$. Conversely, for any rearrangement-invariant space Y on Z with indices equal to $q^{-1}, 2 < q \leqq \infty$, there is a rearrangement-invariant space $\check{Y}$ on T with indices equal to p^{-1} such that $\mathscr{T}$ is bounded from $\check{Y}$ into Y.

Analogous results for other groups are indicated and examples are discussed when X is L^p or a Lorentz space L^{pr}.

By $L^p = L^p(T)$ we denote the usual Lebesgue space on the unit circle T, and by $l^p = l^p(Z)$ the corresponding space on the integers Z. The index conjugate to p will always be denoted by q so that $p^{-1} + q^{-1} = 1$. The Fourier transform $\mathscr{T}$ defined by

$$(\mathscr{T} f)(n) = \hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta ,$$

is a bounded linear operator from L^1 into l^∞ and from L^2 into l^2 so by the M. Riesz-Thorin interpolation theorem ([9], p. 95), $\mathscr{T}$ is bounded also from L^p into l^q whenever $1 < p < 2$. This is the assertion of the classical Hausdorff-Young theorem ([9], p. 101).

Hardy and Littlewood ([5]; [9], p. 109) showed that $\mathscr{T}$ is bounded from $L^p, 1 < p < 2$, into l_p^q, the "weighted" Lebesgue space of all sequences $\{c_n\}$ for which

$$\|c\|_{l_p^q} = \left\{\sum_{-\infty}^{\infty} |c_n| p(1 + |n|)^{p-2}\right\}^{1/p} = \left\{\sum_{-\infty}^{\infty} [(1 + |n|)^{1/q} |c_n|]^p (1 + |n|)^{-1}\right\}^{1/p}$$

is finite; since $l_p^q \subseteqq l^q$, as a simple computation shows, their result improves on that of Hausdorff and Young. A still sharper version, again due to Hardy and Littlewood ([6]; [9], p. 123), is based upon the observation that even the (symmetric) decreasing rearrangement of the sequence $\{\hat{f}(n)\}$ belongs to l_p^q, or, what amounts to the same thing, $\mathscr{T} f$ belongs to the Lorentz space l^{qp} (cf. [3], [4], and [9] for the precise statements and definitions). Thus $\mathscr{T}$ is a bounded