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BOREL SETS OF PROBABILITY MEASURES

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Let $M(X)$ be the collection of probability measures on the Borel sets of a Polish space X. The Borel structure of $M(X)$ generated by the weak* topology is investigated. Various collections of probability measures arising in nonparametric statistics are shown to Borel sets of $M(X)$. Attention is particularly focused on collections arising from restrictions on distribution functions, density functions, and supports of the underlying probability measures.

1. Introduction. Baynesian statisticians assume prior distributions on certain families of probability measures. This amounts to putting a probability measure on a family of probability measures. Now families of probability measures typically arising in statistics are parametrized by some Borel set of Euclidean n-space. In such cases, one has a natural Borel structure or σ-algebra of subsets with which to deal. In nonparametric situations the natural Borel structure is not so obvious. Ideally, one might desire each commonly occurring family of probability measures to be a Borel set of some properly chosen complete separable metric space. Then a prior distribution could be viewed as a probability measure on the entire space which is concentrated on the given Borel set. Our aim is to show that many, if not most, nonparametric families of probability measures are indeed Borel sets of complete separable metric spaces. This advances slightly the cause of nonparametric Baynesian statistics, but does not overcome the more difficult barrier of finding reasonable prior distributions in nonparametric situations.

In our probabilistic model we suppose X to be a complete separable metric space. Let $C(X)$ be the bounded real-valued continuous functions on X under the sup norm topology. Then the collection of probability measures $M(X)$ on the Borel sets of X can be viewed as a subset of the dual of $C(X)$ under the weak* topology. It is well known that $M(X)$ is metrizable as a complete separable metric space with this topology [16]. Our investigations will center on the Borel structure of $M(X)$.

Dubins and Freedman have done the spadework for the subsequent discussion in their basic paper [8]. Section 3 generalizes their analysis of the relationship of distribution and density functions to probability measures. Section 4 explores the connection between a probability measure and its support when the underlying space is no longer compact. Section 5 collects some further examples not considered in