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THE TYPE OF SOME C^* AND W^*-ALGEBRAS ASSOCIATED WITH TRANSFORMATION GROUPS

ELLIOT C. GOOTMAN

Let (G, Z) be a second countable locally compact topological transformation group, $\mathscr{U}(G, Z)$ the associated C^*-algebra and L a certain naturally constructed representation of $\mathscr{U}(G, Z)$ on $L^2(G \times Z, dg \times d\alpha)$, dg being left Haar measure on G and α a quasi-invariant ergodic probability measure on Z. Representations of $\mathscr{U}(G, Z)$ constructed from positive-definite measures on $G \times Z$ are used to prove that $\mathscr{U}(G, Z)$ is type I if and only if all the isotropy subgroups are type I and Z/G is T_0, and, under the assumption of a common central isotropy subgroup, that L has no type I component if α is nontransitive. By means of quasi-unitary algebras, necessary and sufficient conditions are derived for L to be semi-finite under the weaker assumption of a common type I unimodular isotropy subgroup.

After establishing notation and discussing preliminary material in § 2, we prove in § 3 that $\mathscr{U}(G, Z)$ is type I if and only if Z/G is T_0 and all isotropy subgroups are type I. This result, proven by Glimm [9, Theorem 2.2] for the special case in which isotropy subgroups can be chosen "continuously", is not surprising in light of Mackey's Imprimitivity Theorem and the correspondence between representations of $\mathscr{U}(G, Z)$ and systems of imprimitivity based on (G, Z) (see § 2). Our general proof, based on the fact that isotropy subgroups can always be chosen "measurably" [1, Proposition 2.3], follows by construction of a direct integral of certain representations which, by being defined in terms of positive-definite measures, are easily specified and shown to form an integrable family.

In §§ 4 and 5 we consider the type of a W^*-algebra $\mathscr{A}$ constructed via an ergodic quasi-invariant probability measure α on Z (see § 4 for the construction). This algebra was studied by Murray and von Neumann in [14], [15], and [16] for the case of G discrete (see also [4, pp. 127-137]), by Dixmier in [3, §§ 10-12] for the case of G acting freely on Z and by Kallman in [10] for the case in which α is transitive. In § 4 we first show that $\mathscr{A}$ is the von Neumann algebra generated by the representation of $\mathscr{U}(G, Z)$ determined by the positive-definite measure $\delta_e \times d\alpha$ on $G \times Z$. Then assuming that almost all ($d\alpha$) points in Z have the same isotropy subgroup H, we use a direct integral decomposition of $\mathscr{A}$ arising naturally from a consideration of the measure $\delta_e \times d\alpha$ to prove that if α is nontransitive and if H is in