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A NONASSOCIATIVE EXTENSION OF THE CLASS OF DISTRIBUTIVE LATTICES

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Let $Z = \{0, 1, 2\}$ and define two binary operations $\wedge$ and $\vee$ on Z as follows: $0 \wedge 1 = 0, 0 \vee 1 = 1, 1 \wedge 2 = 1, 1 \vee 2 = 2, 2 \wedge 0 = 2, 2 \vee 0 = 2$, both operations are idempotent and commutative. This paper deals with the equational class $\boldsymbol{Z}$ generated by the algebra $\langle Z; \wedge, \vee\rangle$. The class $\boldsymbol{Z}$ contains the class of all distributive lattices and $\boldsymbol{Z}$ is a subclass of the class of weakly associative lattices (*trellis*, T-lattice) in the sense of E. Fried and H. Skala.

The purpose of this paper is to prove that $\boldsymbol{Z}$ shares the most important properties of the class of distributive lattices.

A *tournament* $\langle T; <\rangle$ is a set T with a binary relation $<$ such that for all $a, b \in T$ exactly one of $a = b, a < b$, and $b < a$ holds. Equivalently, a tournament is a directed graph without loops such that exactly one directed edge connects any two distinct points. Just as chains (linearly ordered sets) can be turned into lattices we can define meet and join on a tournament $\langle T; <\rangle$ by the rule:

$$\text{if } x < y, \text{ then } x = x \wedge y = y \wedge x \text{ and } y = x \vee y = y \vee x\,,$$
$$\text{and } x = x \wedge x = x \vee x \text{ for all } x.$$

Since for all $x, y \in T, x \neq y$, we have $x < y$ or $y < x$ the above rule defines $\wedge$ and $\vee$ on T.

Of course, the algebra $\langle T; \wedge, \vee\rangle$ we constructed is not a lattice: neither $\wedge$ nor $\vee$ is associative unless $\langle T; <\rangle$ is a chain, that is, $<$ is transitive. However, as it was observed in E. Fried [5], the two operations are idempotent, commutative; the absorption identities hold and also a weak form of the associative identities.

The smallest example of a nontransitive tournament is the three-element cycle $\langle\{0, 1, 2\}; <\rangle$ in which $0 < 1, 1 < 2$, and $2 < 0$. In the corresponding algebra Z neither $\wedge$ nor $\vee$ is associative.

Z plays the same role for tournaments as the two-element lattice does for distributive lattices. A tournament (algebra) $\langle T; \wedge, \vee\rangle$ is not a chain if and only if it contains Z as a subalgebra.

In this paper we investigate the equational class $\boldsymbol{Z}$ generated by the algebra Z. Observe that $C_2 = \langle\{0, 1\}; \wedge, \vee\rangle$ is a subalgebra of Z, in fact, it is a two-element chain. Therefore, $\boldsymbol{Z}$ contains as a subclass the class $\boldsymbol{D}$ of all distributive lattices. (Indeed, $\boldsymbol{D}$ is generated by C_2.)

The results of this paper can be summarized as follows: many of