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MAXIMAL INVARIANT SUBSPACES OF STRICTLY CYCLIC OPERATOR ALGEBRAS

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A *strictly cyclic* operator algebra $\mathscr{A}$ on a complex Banach space $X(\dim X \geqq 2)$ is a uniformly closed subalgebra of $\mathscr{L}(X)$ such that $\mathscr{A}x = X$ for some x in X. In this paper it is shown that (i) if $\mathscr{A}$ is strictly cyclic and intransitive, then $\mathscr{A}$ has a maximal (proper, closed) invariant subspace and (ii) if $A \in \mathscr{L}(X)$, $A \neq zI$ and $\{A\}'$ (the commutant of A) is strictly cyclic, then A has a maximal hyperinvariant subspace.

1. **Notation and terminology.** Throughout the paper X is a complex Banach space of dimension greater than one and $\mathscr{L}(X)$ is the algebra of continuous linear operators on X. $\mathscr{A}$ will denote a uniformly closed subalgebra of $\mathscr{L}(X)$ which is *strictly cyclic* and x_0 will be a *strictly cyclic vector* for $\mathscr{A}$: that is, $\mathscr{A}x_0 = X$. We do not insist that the identity element I of $\mathscr{L}(X)$ be an element of $\mathscr{A}$.

If $\mathscr{B} \subset \mathscr{L}(X)$, then the *commutant* of $\mathscr{B}$ is $\mathscr{B}' = \{E: E \in \mathscr{L}(X)$ and $EB = BE$ for all B in $\mathscr{B}\}$. We shall use the terminology of "invariant" and "transitive" as follows: if $M \subset X$ and $\mathscr{B} \subset \mathscr{L}(X)$, then (i) M is *invariant* under $\mathscr{B}$ if $\mathscr{B}M = \{Bm: B \in \mathscr{B}$ and $m \in M\} \subset M$, (ii) M is an *invariant subspace* for $\mathscr{B}$ if M is invariant under $\mathscr{B}$ and M is a closed, nontrivial ($\neq \{0\}, X$) linear subspace of X, (iii) $\mathscr{B}$ is *transitive* if $\mathscr{B}$ has no invariant subspace and *intransitive* if $\mathscr{B}$ has an invariant subspace. Further, if $A \in \mathscr{L}(X)$ and $\{A\}'$ is intransitive, then each invariant subspace of $\{A\}'$ is called a *hyperinvariant subspace* of A. Finally an invariant subspace of $\mathscr{B}$ is *maximal* if it is not properly contained in another invariant subspace of $\mathscr{B}$.

2. **Introduction.** Strictly cyclic operator algebras have been studied by A. Lambert, D. A. Herrero, and the auther of this paper. (See for example [2]-[6].) One of the major results in [2, Theorem 3.8], [3, Theorem 2], and [6, Theorem 4.5] is that a transitive subalgebra of $\mathscr{L}(X)$ containing a strictly cyclic algebra is necessarily strongly dense in $\mathscr{L}(X)$. In each of three developments the following is a key lemma: The only dense linear manifold invariant under a strictly cyclic subalgebra of $\mathscr{L}(X)$ is X. In Lemma 1 we shall present a generalization of this lemma which will be useful in the study of maximal invariant subspaces and noncyclic vectors of a strictly cyclic algebra $\mathscr{A}$.

LEMMA 1. *If M is invariant under $\mathscr{A}$ and $x_0 \in \bar{M}$, then $M = X$.*