PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

THE NUMBER OF VECTORS JOINTLY ANNIHILATED BY TWO REAL QUADRATIC FORMS DETERMINES THE INERTIA OF MATRICES IN THE ASSOCIATED PENCIL

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Pencils of real symmetric matrices and their associated quadratic forms are interrelated. It is well known that a pencil contains a definite matrix iff the associated quadratic forms do not vanish simultaneously, provided the matrices have dimension $n \geqq 3$. This knowledge is extended here to yield the following for nonsingular pairs of real symmetric matrices of dimension $n \geqq 3$:

(I) The pencil $P(S, T)$ contains a semidefinite, but no definite matrix iff the maximal number l of lin. ind. vectors simultaneously annihilated by the associated quadratic forms lies between 1 and $n-1$ and certain conditions on S and T hold if $l = n-1$.

(II) The pencil $P(S, T)$ contains only indefinite matrices iff $n-1 \leqq l \leqq n$ with other (complementary to the above) conditions holding if $l = n-1$.

First we introduce the relevant notation for a pair of real symmetric (r.s.) matrices S and T of the same dimension n:

DEFINITION 1. (a) The pencil $P(S, T) = \{aS + bT \mid a, b \in \boldsymbol{R}\}$ is a *d-pencil* if $P(S, T)$ contains a definite matrix.

(b) $P(S, T)$ is a *s.d. pencil* if $P(S, T)$ contains a nonzero semidefinite, but no definite matrix.

(c) $P(S, T)$ is an *i-pencil* if $P(S, T)$ contains only indefinite matrices, except for the zero matrix.

NOTATION. We denote by Q_S the set $\{x \in \boldsymbol{R}^n \mid x'Sx = 0\}$.

DEFINITION 2. A pair of r.s. matrices S and T is called a *nonsingular pair* if S is nonsingular.

This is our main result:

MAIN THEOREM. *For a pair of* r.s. *matrices S and T of dimension $n \geqq 3$ let $l = \max\{k \mid$ there exist k* lin. ind. *vectors in $Q_S \cap Q_T\}$. Then we have:*

(a) *$P(S, T)$ is a d-pencil iff $l = 0$, and for a nonsingular pair S, T:*

(b) *$P(S, T)$ is a s.d. pencil if and only if $1 \leqq l \leqq n-1$ and*