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ON FINDING THE DISTRIBUTION FUNCTION FOR AN ORTHOGONAL POLYNOMIAL SET

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Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be real sequences with $b_n > 0$ and $\{b_n\}_{n=0}^{\infty}$ bounded. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials satisfying the recurrence formula

$$\tag{1.1} \begin{cases} xP_n(x) = b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x) & (n \geqq 0) \\ P_{-1}(x) = 0 \qquad P_0(x) = 1. \end{cases}$$

Then there is a substantially unique distribution function $\psi(t)$ with respect to which the $P_n(x)$ are orthogonal. That is,

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)d\psi(x) = K_n\delta_{n,m} \qquad (n, m \geqq 0),$$

where $K_n \neq 0$ and $\delta_{n,m}$ is the kronecker delta. This paper gives a method of constructing $\psi(x)$ for the case $\lim_{n\to\infty} b_{2n} = 0$, $\lim_{n\to\infty} b_{2n+1} = b < \infty$, the set of limit points of $\{a_n\}_{n=1}^{\infty}$ equals $\{-\alpha, \alpha\}$ and $\lim_{n\to\infty}\{a_{2n} + a_{2n+1}\} = 0$. The same method can be used in the case $\lim_{n\to\infty} b_n = 0$ and the set of limit points of $\{a_n\}_{n=0}^{\infty}$ is bounded and finite in number.

This continues the investigation started by Dickinson, Pollak, and Wannier [3] in which they studied the distribution function under the assumption $a_n = 0$ and $\Sigma\, b_n < \infty$. Goldberg [4] extended their results by considering the case $a_n = 0$ and $\lim_{n\to\infty} b_n = 0$. Finally, Maki [5] showed how to construct the distribution function when $\lim_{n\to\infty} b_n = 0$ and the set of limit points of $\{a_n\}_{n=0}^{\infty}$ are bounded and finite in number. In all these cases their approach was to study the continued fraction

$$\tag{1.2} K(z) = \frac{1\ |}{|\ z - a_0} - \frac{b_0^2\ |}{|\ z - a_1} - \frac{b_1^2\ |}{|\ z - a_2}\cdots,$$

where $\{b_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ consist of the same numbers as given in (1.1).

Our approach is different from that of the above mentioned authors. If $S(\psi)$ denotes the spectrum of ψ, i.e., the set $\{\lambda \mid \psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) > 0$ for all $\varepsilon > 0\}$, then, in our case, we will show from the properties of the sequences $\{a_n\}$ and $\{b_n\}$ how to find the derived set of $S(\psi)$ and that the $S(\psi)$ consists of a denumerable set of points.

To prove our results we make use of the following theorem due to M. Krein ([1], p. 230-231).

THEOREM 1.1. *The polynomial set defined by* (1.1) *is associated with a determined Hamburger moment problem with solution ψ, such that $S(\psi)$ is bounded and the set of limit points of $S(\psi)$ is contained*