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GLOBAL PROPERTIES OF RATIONAL AND LOGARITHMICO-RATIONAL MINIMAL SURFACES

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In this paper, rational and logarithmico-rational minimal surfaces are defined and some of their properties are investigated. In particular, it is shown—perhaps somewhat surprisingly—that the fundamental theorem of algebra, suitably formulated, holds for these surfaces.

1. Introduction. Let $f(w)$ be a rational complex function, of degree m, of the complex variable w; that is, let $f(w)$ admit a representation of the form

$$f(w) = \frac{p(w)}{q(w)} ,$$

where $p(w)$ and $q(w)$ are relatively prime complex polynomial functions:

$$[p(w), q(w)] = 1 ;$$

and let

$$\deg f(w) = \max [\deg p(w), \deg q(w)] = m .$$

It is well known (see, for instance, [1, p. 31]) that, with multiple values suitably counted, for any such rational function of degree m, $m \geqq 1$, the equation $\zeta = f(w)$ maps the closed single-sheeted w-plane onto the closed m-sheeted ζ-plane. That is, each ζ-value, including ∞, is taken on exactly m times (counting multiplicities) as w ranges once over the closed complex plane. This is a rather immediate consequence of the fundamental theorem of algebra.

Now for $n \geqq 3$, a (two-dimensional) minimal surface S in n-dimensional Euclidean space E_n has n-dimensional measure 0, and accordingly the foregoing result concerning complex rational functions $f(w)$ could hardly be expected to extend directly to minimal-surface theory. It is nevertheless one purpose of this paper to show (see §10) that the result, *when suitably formulated,* does indeed extend precisely to minimal surfaces.

Since, as we shall see in §4, the map $\zeta = f(w)$ given by a rational complex function $f(w)$ is a special rational minimal surface, it therefore follows in particular that the fundamental theorem of algebra concerning *plane* maps can be given a formulation relative to the points of any containing n-dimensional Euclidean space!