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THE GARABEDIAN FUNCTION OF AN ARBITRARY COMPACT SET

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If the outer boundary of the compact plane set E is the union of finitely many disjoint analytic Jordan curves, the Garabedian function of E is a familiar object. J. Garnett and S. Y. Havinson have each asked whether the Garabedian functions of a decreasing sequence of such sets must converge. The present paper shows that they do converge. This fact leads to a natural definition of the Garabedian function of an arbitrary compact plane set. As an intermediate step, an approximate formula is obtained for the analytic capacity of the union of a compact set E and a small disc not intersecting E.

1. Prerequisites and notation. Good introductions to Analytic Capacity are given in [2], pp. 1–26, and [1], Ch. 8; and so we shall give only a brief outline.

$\boldsymbol{C}$ denotes the complex plane. S^2 denotes the extended complex plane with its usual topology. $D(z; r)$ denotes the closed disc with centre z and radius r.

Let E be a compact subset of $\boldsymbol{C}$. $\Omega(E)$ denotes the component of $S^2\backslash E$ containing ∞. The *outer boundary* of E is the boundary $\partial\Omega(E)$ of $\Omega(E)$. The *analytic capacity* of E is:

$$\gamma(E) = \sup\{|g'(\infty)|: g \text{ analytic on } \Omega(E),\ |g| < 1\}\ .$$

This supremum is attained by a unique function, the *Ahlfors function* of E ([1], p. 197).

$\mathscr{S}$ will denote the class of all compact plane sets whose outer boundary is the union of finitely many pairwise disjoint analytic Jordan curves. Let $E \in \mathscr{S}$, and write $\Omega = \Omega(E)$. The *Hardy space* $H^p(\Omega)$ $(0 < p < \infty)$ is the space of all analytic functions g on Ω such that there exists a harmonic function u on Ω with $|g|^p < u$. If $g \in H^p(\Omega)$ then g has a finite nontangential limit $g(z)$ at almost every point $z \in \partial\Omega$. $H^2(\Omega)$ is a separable Hilbert space with the inner product:

$$(g, h) = \int_{\partial\Omega} g(z)h(z)^*ds \quad (g, h \in H^2(\Omega))\ .$$

If $\zeta \in \Omega$ there is a unique function $K(z, \zeta)$ in $H^2(\Omega)$, the *Szegö kernel function*, such that:

$$g(\zeta) = \int_{\partial\Omega} g(z)K(z, \zeta)^*ds \quad (g \in H^2(\Omega))\ .$$

$K(z, \zeta)$ is the inner product between the functionals on $H^2(\Omega)$ given