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SUBALGEBRAS OF FINITE CODIMENSION IN THE ALGEBRA OF ANALYTIC FUNCTIONS ON A RIEMANN SURFACE

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Let R be a finite open Riemann surface with boundary Γ. We set $\bar{R} = R \cup \Gamma$ and let $A(R)$ denote the algebra of functions which are continuous on $\bar{R}$ and analytic on R. Suppose A is a uniform algebra contained in $A(R)$. The main result of this paper shows that if A contains a function F which is analytic in a neighborhood of $\bar{R}$ and which maps $\bar{R}$ in a n-to-one manner (counting multiplicity) onto $\{z: |z| \leqq 1\}$, then A has finite codimension in $A(R)$.

We say that A is a uniform algebra on $\bar{R}$ if A is a uniformly closed subalgebra of the complex-valued continuous functions on $\bar{R}$ which separates points of $\bar{R}$ and contains the constant functions. If A is contained in $A(R)$, then we say A has finite codimension in $A(R)$ if $A(R)/A$ is a finite dimensional vector space over $\boldsymbol{C}$. A reference for uniform algebras is Gamelin [2].

Let U be the open unit disk in $\boldsymbol{C}$. We call F an unimodular function if F is analytic in a neighborhood of $\bar{R}$ and maps $\bar{R}$ onto $\bar{U}$ so that F is n-to-one if we count the multiplicity of F where dF vanishes. If T is the unit circle, then F maps Γ onto T. The existence of such a function was first proved by Ahlfors [1]. Later, Royden [4] gave another proof of this result.

1. Main results. Let A be a uniform algebra on $\bar{R}$ which is contained in $A(R)$. If $J = \{f \in A(R): fA(R) \subset A\}$, then J is a closed ideal in $A(R)$ and J is contained in A.

LEMMA. *Let $F \in A$ be an unimodular function of order n. If $\zeta_1 \in \bar{R}$ is such that $F^{-1}(F(\zeta_1))$ consists of n distinct points, then there is $G \in J$ such that $G(\zeta_1) \neq 0$.*

Proof. Since A separates points on $\bar{R}$, there is $g \in A$ such that g separates $F^{-1}(F(\zeta_1))$. If $z_1 \in \bar{R}$, let $F^{-}(F(z_1)) = \{z_1, z_2, \cdots, z_n\}$ (perhaps with repetitions) and let $f \in A(R)$.

Define $Q(u) = f(z_1)\{u - g(z_2)\}\{u - g(z_3)\} \cdots \{u - g(z_n)\} + f(z_2)\{u - g(z_1)\}\{u - g(z_3)\} \cdots \{u - g(z_n)\} + \cdots + f(z_n)\{u - g(z_1)\}\{u - g(z_2)\} \cdots \{u - g(z_{n-1})\}$ (cf. [5], p. 290). Then $Q(u)$ is a polynomial in u of the form $Q(u) = \alpha_{n-1}(z_1, \cdots, z_n)u^{n-1} + \alpha_{n-2}(z_1, \cdots, z_n)u^{n-2} + \cdots + \alpha_0(z_1, \cdots, z_n)$. The coefficients α_j are symmetric functions in $z_1, \cdots, z_n$. Hence, if