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ON THE EIGENVALUES OF A SECOND ORDER ELLIPTIC OPERATOR IN AN UNBOUNDED DOMAIN

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Let E be an open set in R^n which satisfies the "narrowness at infinity" condition:

$$\operatorname{meas}(E \cap \{x \in R^n : a \leqq |x| < a+1\}) \leqq \operatorname{const}(a+1)^{-\beta},$$

for all $a > 0$ and some $\beta > 0$. It is known that a uniformly strongly elliptic self-adjoint partial differential operator, on such a set E, has a discrete spectrum of eigenvalues $\{\lambda_j\}$. This paper is concerned with the growth rate of the function

$$N(\lambda) = \sum_{\lambda_n \leqq \lambda} 1 .$$

The main result of the paper is to give an upper bound for $N(\lambda)$. This upper bound will be a function of the β from the "narrowness" condition.

An unbounded open set E in Euclidean n-space R^n is said to be quasi-bounded if the points $x \in E$ with $|x|$ large are near the boundary ∂E:

$$\lim_{x \to \infty, x \in E} \operatorname{dist}(x, \partial E) = 0 .$$

Let T be the $L_2(E)$-realization of the uniformly strongly elliptic second order partial differential operator $a(x, D)$ with zero Dirichlet boundary conditions:

$$a(x, D) = -\sum_{|\alpha| \leqq 2} a_\alpha(x) D^\alpha , \qquad D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} ,$$

$$|\alpha| = |\alpha_1| + \cdots + |\alpha_n| ,$$

$$a_0(x, \xi) \geqq \operatorname{const} |\xi|^{2m} , \quad x \in R^n , \quad \xi \in R^n$$

where $a_0(x, \xi)$ is the principle part of $a(x, \xi)$; the coefficients $a_\alpha(x)$ are infinitely differentiable bounded real functions in R^n; $a(x, D)$ is formally self-adjoint;

$$\mathscr{D}(T) = H_0^1(E) \cap \{f \in L_2(E) : a(x, D) f \in L_2(E)\}$$

$$Tf = a(x, D) f , \qquad f \in \mathscr{D}(T) ,$$

where $H_0^1(E)$ is the standard Sobolev space. If E is quasi-bounded and satisfies some additional smoothness conditions, that it is known, Clark [4] and Adams [1], that T has a compact resolvent, and thus a discrete spectrum, consisting of eigenvalues λ_j satisfying