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NORMED KÖTHE SPACES AS INTERMEDIATE SPACES OF L_1 AND L_∞

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Let (Δ, Σ, μ) be a totally σ-finite measure space and let $M(\Delta)$ be the set of all complex-valued μ-measurable functions on Δ. This paper is concerned with determining whether certain classes of normed Köthe spaces (Banach function spaces) are intermediate spaces of $L_1 = L_1(\mu)$ and $L_\infty = L_\infty(\mu)$. It is proven that $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are associate Orlicz spaces and that for every nontrivial Young's function Φ there is an equivalent Young's function Φ_1 such that the Orlicz space $L_{M\Phi_1}$ is an intermediate space of L_1 and L_∞. The notion of a universal Köthe space is presented and it is proven that if Λ is a universal Köthe space then $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$. Furthermore, if Λ is normed, in particular $\Lambda = L_\rho$, then there is an equivalent universally rearrangement invariant norm ρ_1 for which L_{ρ_1} is an intermediate space of L_1 and L_∞.

1. Introduction. Let X_1 and X_2 be two Banach spaces contained in a linear Hausdorff space Y such that the injection of $X_i (i = 1, 2)$ into Y is continuous. Denote the norm of X_i by $\|\cdot\|_i$. The space $X_1 \cap X_2$ is the set of all elements which are in both X_1 and X_2, and the space $X_1 + X_2$ is the set of all $f \in Y$ of the form $f = f_1 + f_2$ with $f_1 \in X_1$ and $f_2 \in X_2$. The spaces $X_1 \cap X_2$ and $X_1 + X_2$ are Banach spaces under the norms $\|f\|_{X_1 \cap X_2} = \max\{\|f\|_1, \|f\|_2\}$ and $\|f\|_{X_1+X_2} = \inf\{\|f_1\|_1 + \|f_2\|_2 : f = f_1 + f_2, f_i \in X_i\}$ (see [1, p. 165, Prop. 3.2.1]). A Banach space $X \subset Y$ satisfying $X_1 \cap X_2 \subset X \subset X_1 + X_2$ and $\|f\|_{X_1+X_2} \leqq \|f\|_X \leqq \|f\|_{X_1 \cap X_2}$ is called an *intermediate space* of X_1 and X_2.

Much work has been done on intermediate spaces and the related topic of interpolation theory. (See [1], [2], [12].) In particular, it has been shown that the Lebesgue spaces L_p and the Lorentz spaces L_{pq} ([6] and [7]) are intermediate spaces of L_1 and L_∞. In this paper we investigate what other classes of normed Köthe spaces are intermediate spaces of L_1 and L_∞. In §7 we introduce the notion of a universal Köthe space, which we prove to be equivalent to Luxemburg's notion of a universally rearrangement invariant Köthe space [9]. We have been able to show that if Λ is a universal Köthe space, then $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$. Furthermore, if Λ is normed, in particular $\Lambda = L_\rho$, then there is an equivalent norm ρ_1 which is universally rearrangement invariant and L_{ρ_1} is an intermediate space of L_1 and L_∞.

Section 2 contains preliminaries and §3 deals with Orlicz spaces. We show that $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are Orlicz spaces and prove that they are associate Orlicz spaces. It is shown that for any nontrivial