PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

A FIXED POINT THEOREM FOR k-SET-CONTRACTIONS DEFINED IN A CONE

JUAN A. GATICA AND W. A. KIRK

Let X be a Banach space and H a solid closed cone in X with interior H^0. Suppose B is a bounded open set in X containing the origin. For $G = B \cap H^0$, let $\partial_H \bar{G}$ denote the relative boundary of the closure $\bar{G}$ of G in H. In this paper mappings $T: \bar{G} \to H$ are considered where T is a k-set-contraction, $k < 1$. It is shown for such mappings that if $(I - tT)(G)$ is open, $t \in [0, 1]$, and if T satisfies (i) $Tx \neq \lambda x$ for all $x \in \partial_H \bar{G}$ and $\lambda > 1$, then T has a fixed point in $\bar{G}$. In the special case when T is a contraction mapping, $(I - tT)(G)$ is always open and boundedness of B can be dispensed with.

The Leray-Schauder boundary condition (i) is an assumption which in particular holds *for convex* G if $T: \partial_H \bar{G} \to \bar{G}$, or even more generally if T is 'inward' in the sense of Halpern and Bergman [7] (cf. also, Vidossich [15] (Theorem 5(ii)) for an equivalent condition on $f = I - T$). Conditions similar to (i) have been imposed by several authors recently in proving fixed point theorems in functional analysis, although, as we note in more detail below, it is usually assumed that the origin is an interior point of the domain of T, with the condition $Tx \neq \lambda x$, $\lambda > 1$, required of all x in the boundary of this domain.

We are concerned here with the "k-set-contractions", $k < 1$, a class of mappings which includes not only the usual "contraction mappings" (mappings $U: D \to X$ satisfying for some $\alpha < 1$, $\|Ux - Uy\| \leqq \alpha \|x - y\|$, $x, y \in D$), but also mappings of the form $T = U + C$ with U a contraction mapping and C compact. This class is defined by Kuratowski [9] as follows: For a bounded subset A of X define the measure of noncompactness, $\gamma(A)$, of A by $\gamma(A) = \text{g.l.b.}\ \{d > 0$: there exists a finite number of sets $S_1, \cdots, S_n$ such that $A \subset \bigcup_{i=1}^n S_i$ and diam $S_i \leqq d$, $i = 1, \cdots, n\}$. A continuous mapping $T: D \to X$, $D \subset X$, is called a k-set-contraction if there is a fixed constant $k \geqq 0$ such that $\gamma(T(A)) \leqq k\gamma(A)$ for all bounded $A \subset D$. There has been intensive study of these mappings recently including, notably, Nussbaum's development [10] of a theory of topological degree for them.

With H and G as above, we prove in this paper that if $T: \bar{G} \to H$ is a k-set-contraction, $k < 1$, satisfying (i) on $\partial_H \bar{G}$, and if $(I - tT)(G)$ is open, $t \in [0, 1]$, then T has a fixed point in $\bar{G}$. This theorem is specifically related to a number of recent results; for example, Nussbaum has proved [10] that if D is a bounded closed and convex