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STRONGLY UNIQUE BEST APPROXIMATES TO A FUNCTION ON A SET, AND A FINITE SUBSET THEREOF

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Let X be a compact Hausdorff space and let $C(X)$ denote the space of continuous real valued functions defined on X, normed by the supremum norm $\|f\| = \max_{x \in X} |f(x)|$. Let M be a finite dimensional subspace of $C(X)$. This note examines the problem of whether every best (unique best, strongly unique best) approximate to f on X is also a best (respectively: unique best, strongly unique best) approximate to f on some finite subset of X. Appropriate converse results are also considered.

The Kolmogorov criterion for best approximates shows that $\pi \in M$ is a best approximate to f on X if and only if it is a best approximate to f on a finite subset of

$$E_\pi = \{x \in X: |f(x) - \pi(x)| = \|f - \pi\|\} .$$

Example 1 shows that the corresponding result does not hold for unique best approximates. It can easily be shown that when π is a strongly unique best approximate to f in $C[a, b]$ from a Haar subspace then there is a finite subset A of $[a, b]$ such that π is a strongly unique best approximate to f on A. In Theorem 2 the latter result is extended to an arbitrary finite dimensional subspace M of $C(X)$ and in Theorem 3 a converse is proven in this general setting.

The second algorithm of Remez [11] is an important method for the computation of the best approximate to a function f in $C[a, b]$ from a finite dimensional Haar subspace. This algorithm depends on the fact that a best approximate to f on $[a, b]$ is a best approximate to f on some finite subset of $[a, b]$. (One can think of the algorithm as a search for this subset.) In fact, the proof of the convergence of the algorithm given by E. W. Cheney [3] indicates that the algorithm depends more precisely on the facts that the best approximate π to f on $[a, b]$ is strongly unique and that π is also a strongly unique best approximate to f on some finite subset of $[a, b]$.

It would also be natural to consider in $L^p[a, b]$ for $1 \leqq p < \infty$ the relationship between strongly unique best approximates on $[a, b]$ and on finite subsets of $[a, b]$. However, D. E. Wulbert ([15], [16]) has shown that strong unicity does not occur (nontrivially) in any smooth space and $L^p[a, b]$ for $1 \leqq p < \infty$ is smooth. In the last section a different proof of Wulbert's result is given because the