PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 2, 1974

FUNCTIONAL REPRESENTATION OF ALGEBRAIC INTERVALS

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Motivated by some examples from the study of axiomatic convexity, we define a class of objects (in real algebras with 1) whose algebraic properties mimic those of the unit interval. These objects, called intervals, have quite a bit of structure in themselves. In particular, in a Banach algebra a compact interval must be finite dimensional. Even more striking is the main result which shows that any interval satisfying a very modest boundedness condition is commutative and can be represented by continuous functions from a compact Hausdorff space into the unit interval. This leads to a number of corollaries in analysis and topology.

THEOREM 1. *Let M be a linear space of real functions on a set X, and suppose that any function in M is the difference of two nonnegative functions in M. If S and T are linear maps from M to M such that*

$$0 \leqq S(f) \leqq f \quad and \quad 0 \leqq T(f) \leqq f$$

for all nonnegative f in M, then S and T commute.

THEOREM 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that* $\limsup (a_n)^{1/n} = \infty$. *Suppose P is the smallest set of real polynomials containing 0 and $a_n x^n$, for all n, and satisfying*

(1) *$p(x)q(x) \in P$ if p and q are in P and*

(2) *$-p(x) + 1 \in P$ if p is in P.*

Then any real polynomial r such that $0 < r(0) < 1$ is a convex combination of polynomials in P.

These two seemingly disparate results, as well as a representation theorem by Stone for partially ordered algebras, follow as natural corollaries of a general theory to be developed in this paper. This theory had its origins in an attempt to investigate more general notions of convexity in real linear spaces, and that setting still seems to provide the best starting place of the development.

DEFINITION. If A is a ring with 1, then an *interval* in A is a subset I of A such that

(1) $0 \in I$,

(2) $1 - x \in I$ if $X \in I$, and