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STRONGLY SEMISIMPLE ABELIAN GROUPS

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For an abelian group G and a ring R, R is a ring on G if the additive group of R is isomorphic to G. G is nil if the only ring R on G is the zero ring, $R^2 = \{0\}$. G is radical if there is a nonzero ring on G that is radical in the Jacobson sense. Otherwise, G is antiradical. G is semisimple if there is some (Jacobson) semisimple ring on G, and G is strongly semisimple if G is nonnil and every nonzero ring on G is semisimple. It is shown that the only strongly semisimple torsion groups are cyclic of prime order, and that no mixed group is strongly semisimple. The torsion free rank one strongly semisimple groups are characterized in terms of their type, and it is shown that the strongly semisimple and antiradical rank one groups coincide. For torsion free groups it is shown that the property of being strongly semisimple is invariant under quasi-isomorphism and that a strongly semisimple group is strongly indecomposable. Further, for a strongly indecomposable torsion free group G of finite rank, the following are equivalent: (a) G is semisimple, (b) G is strongly semisimple, (c) $G \cong R^+$ where R is a full subring of an algebraic number field K such that $[K,Q] = \text{rank } G$ where Q is the field of rational numbers and $R \doteq J_\pi$, where π is either empty or an infinite set of primes in K, $(d) G$ is nonnil and antiradical.

Introduction. In [4], F. Haimo considered the problem of characterizing those abelian groups G that are the additive groups of nontrivial radical rings, where the radical under consideration is the Jacobson radical. It was observed by the present authors that for several classes of groups, those groups G that did not support nontrivial radical rings (antiradical groups) satisfied a much stronger condition, namely, that every nontrivial ring on G is semisimple (strongly semisimple groups). This suggested the problem of identifying classes of groups for which the antiradical and strongly semisimple groups coincide, and the problem of characterizing strongly semisimple groups.

Section 1 contains the basic definitions. The case of torsion and mixed groups is disposed of in §2 where it is shown that the only strongly semisimple torsion groups are the cyclic groups of prime order, and that no mixed group is strongly semisimple. In §3, the torsion free rank one strongly semisimple groups are characterized in terms of their type, and it is shown that the strongly semisimple and antiradical groups coincide. In §4, it is shown that the property